Exercise 3

From $\nabla \cdot \mathbf{B} = 0$ it follows that there exists a vector function $\mathbf{A}$ such that $\nabla \times \mathbf{A} = \mathbf{B}$. This is a well-known fact in vector analysis; see [EP], [Kr], [Sg1].

(a) Show from Maxwell's equations that there also exists a scalar function u such that $-\nabla u = \mathbf{E} + c^{-1}\partial \mathbf{A}/\partial t$.

(b) Deduce from (2) that

$$-c^{-1}\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} - \Delta u = 4\pi\rho$$

$$\text{and} \qquad \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} + \nabla\left(\nabla \cdot \mathbf{A} + c^{-1}\frac{\partial u}{\partial t}\right) = \frac{4\pi}{c}\mathbf{J}.$$

(c) Show that if $\mathbf{A}$ is replaced by $\mathbf{A} + \nabla\lambda$ and u by $u - (1/c)\partial\lambda/\partial t$, then the equations in parts (a) and (b) are still valid for the new $\mathbf{A}$ and the new u. This property is called *gauge invariance.*

(d) Show that the scalar function λ may be chosen so that the new $\mathbf{A}$ and the new u satisfy $\nabla \cdot \mathbf{A} + c^{-1}\partial u/\partial t = 0$.

(e) Conclude that the new potentials satisfy

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = 4\pi\rho \qquad \text{and} \qquad \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} = \frac{4\pi}{c}\mathbf{J}.$$

$\mathbf{A}$ is called the vector potential and u the scalar potential. The equations in part (e) are inhomogeneous wave equations. The transformation in part (c) is the simplest example of a *gauge transformation.*

Solution

Part (a)

Maxwell's inhomogeneous equations are

$$\text{(I)} \quad \frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - 4\pi\mathbf{J}$$

$$\text{(II)} \quad \frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$

$$\text{(III)} \quad \nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\text{(IV)} \quad \nabla \cdot \mathbf{B} = 0.$$

Equation (IV) can be satisfied automatically by introducing a magnetic potential function partly defined by $\nabla \times \mathbf{A} = \mathbf{B}$. The reason this works is because the divergence of any curl is always zero. Substitute this formula into equation (II).

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -c\nabla \times \mathbf{E}$$

$$\nabla \times \frac{\partial \mathbf{A}}{\partial t} = -c\nabla \times \mathbf{E}$$

Bring both terms to the left side.

$$\nabla \times \frac{\partial \mathbf{A}}{\partial t} + c\nabla \times \mathbf{E} = \mathbf{0}$$

Divide both sides by c.

$$\frac{1}{c}\nabla \times \frac{\partial \mathbf{A}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}$$

$$\nabla \times \left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E}\right) = \mathbf{0}$$

This equation can be satisfied automatically by introducing another potential function $-u$ defined by

$$\nabla(-u) = \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E},$$

or

$$-\nabla u = \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}.$$

The reason this works is because the curl of any gradient of a scalar function is always zero.

Part (b)

Take the divergence of both sides of the result from part (a).

$$\nabla \cdot (-\nabla u) = \nabla \cdot \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right)$$

$$-\nabla^2 u = \nabla \cdot \mathbf{E} + \frac{1}{c}\nabla \cdot \frac{\partial \mathbf{A}}{\partial t}$$

Substitute equation (III) here for the divergence of $\mathbf{E}$.

$$-\Delta u = 4\pi\rho + \frac{1}{c}\nabla \cdot \frac{\partial \mathbf{A}}{\partial t}$$

Therefore, equations (II), (III), and (IV) are encapsulated by

$$\boxed{-\frac{1}{c}\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} - \Delta u = 4\pi\rho.}$$

The aim now is to get the second desired result by starting from equation (I).

$$\frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - 4\pi\mathbf{J}$$

Replace $\mathbf{B}$ with $\nabla \times \mathbf{A}$ and use the result of part (a) to eliminate $\mathbf{E}$.

$$\frac{\partial}{\partial t}\left(-\nabla u - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = c\nabla \times (\nabla \times \mathbf{A}) - 4\pi\mathbf{J}$$

$$-\nabla\left(\frac{\partial u}{\partial t}\right) - \frac{1}{c}\frac{\partial^2 \mathbf{A}}{\partial t^2} = c\nabla \times (\nabla \times \mathbf{A}) - 4\pi\mathbf{J}$$

Divide both sides by c.

$$-\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=\nabla\times(\nabla\times\mathbf{A})-\frac{4\pi}{c}\mathbf{J}$$

Solve for $(4\pi/c)\mathbf{J}$ and simplify the curl.

$$\begin{aligned}
\frac{4\pi}{c}\mathbf{J}&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\nabla\times(\nabla\times\mathbf{A})\\
&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\left(\sum_{i=1}^3\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left[\left(\sum_{j=1}^3\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\right)\times\left(\sum_{k=1}^3\boldsymbol{\delta}_kA_k\right)\right]\\
&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\left(\sum_{i=1}^3\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left[\sum_{j=1}^3\sum_{k=1}^3(\boldsymbol{\delta}_j\times\boldsymbol{\delta}_k)\frac{\partial A_k}{\partial x_j}\right]\\
&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\left(\sum_{i=1}^3\boldsymbol{\delta}_i\frac{\partial}{\partial x_i}\right)\times\left(\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\boldsymbol{\delta}_l\varepsilon_{jkl}\frac{\partial A_k}{\partial x_j}\right)\\
&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3(\boldsymbol{\delta}_i\times\boldsymbol{\delta}_l)\varepsilon_{jkl}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j}\\
&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\sum_{m=1}^3\boldsymbol{\delta}_m\varepsilon_{ilm}\varepsilon_{jkl}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j}\\
&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\sum_{l=1}^3\sum_{m=1}^3\boldsymbol{\delta}_m\varepsilon_{mil}\varepsilon_{jkl}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j}\\
&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\sum_{m=1}^3\boldsymbol{\delta}_m(\delta_{mj}\delta_{ik}-\delta_{mk}\delta_{ij})\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j}\\
&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\sum_{m=1}^3\boldsymbol{\delta}_m\delta_{mj}\delta_{ik}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j}-\sum_{i=1}^3\sum_{j=1}^3\sum_{k=1}^3\sum_{m=1}^3\boldsymbol{\delta}_m\delta_{mk}\delta_{ij}\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_j}\\
&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\sum_{i=1}^3\sum_{j=1}^3\boldsymbol{\delta}_j\frac{\partial}{\partial x_i}\frac{\partial A_i}{\partial x_j}-\sum_{i=1}^3\sum_{k=1}^3\boldsymbol{\delta}_k\frac{\partial}{\partial x_i}\frac{\partial A_k}{\partial x_i}\\
&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\sum_{j=1}^3\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\left(\sum_{i=1}^3\frac{\partial A_i}{\partial x_i}\right)-\sum_{i=1}^3\frac{\partial^2}{\partial x_i^2}\left(\sum_{k=1}^3\boldsymbol{\delta}_kA_k\right)\\
&=\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)+\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}\\
&=\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\Delta\mathbf{A}+\nabla(\nabla\cdot\mathbf{A})+\frac{1}{c}\nabla\left(\frac{\partial u}{\partial t}\right)
\end{aligned}$$

Therefore, equation (I) becomes

$$\boxed{\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\Delta\mathbf{A}+\nabla\left(\nabla\cdot\mathbf{A}+\frac{1}{c}\frac{\partial u}{\partial t}\right)=\frac{4\pi}{c}\mathbf{J}.}$$

Part (c)

Replace $\mathbf{A}$ with $\mathbf{A}+\nabla\lambda$ and replace u with $u-(1/c)\partial\lambda/\partial t$ in the result of part (a).

$$-\nabla u = \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$-\nabla\left(u - \frac{1}{c}\frac{\partial\lambda}{\partial t}\right) = \mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}(\mathbf{A}+\nabla\lambda)$$

$$-\nabla u + \frac{1}{c}\nabla\left(\frac{\partial\lambda}{\partial t}\right) = \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \frac{1}{c}\frac{\partial}{\partial t}(\nabla\lambda)$$

$$-\nabla u + \cancel{\frac{1}{c}\nabla\left(\frac{\partial\lambda}{\partial t}\right)} = \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} + \cancel{\frac{1}{c}\nabla\left(\frac{\partial\lambda}{\partial t}\right)}$$

$$-\nabla u = \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

Now replace $\mathbf{A}$ with $\mathbf{A}+\nabla\lambda$ and replace u with $u-(1/c)\partial\lambda/\partial t$ in the first result of part (b).

$$-\frac{1}{c}\nabla\cdot\frac{\partial \mathbf{A}}{\partial t} - \Delta u = 4\pi\rho$$

$$-\frac{1}{c}\nabla\cdot\frac{\partial}{\partial t}(\mathbf{A}+\nabla\lambda) - \Delta\left(u - \frac{1}{c}\frac{\partial\lambda}{\partial t}\right) = 4\pi\rho$$

$$-\frac{1}{c}\nabla\cdot\left[\frac{\partial \mathbf{A}}{\partial t} + \nabla\left(\frac{\partial\lambda}{\partial t}\right)\right] - \Delta u + \frac{1}{c}\Delta\left(\frac{\partial\lambda}{\partial t}\right) = 4\pi\rho$$

$$-\frac{1}{c}\nabla\cdot\frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c}\nabla\cdot\nabla\left(\frac{\partial\lambda}{\partial t}\right) - \Delta u + \frac{1}{c}\Delta\left(\frac{\partial\lambda}{\partial t}\right) = 4\pi\rho$$

$$-\frac{1}{c}\nabla\cdot\frac{\partial \mathbf{A}}{\partial t} - \cancel{\frac{1}{c}\Delta\left(\frac{\partial\lambda}{\partial t}\right)} - \Delta u + \cancel{\frac{1}{c}\Delta\left(\frac{\partial\lambda}{\partial t}\right)} = 4\pi\rho$$

$$-\frac{1}{c}\nabla\cdot\frac{\partial \mathbf{A}}{\partial t} - \Delta u = 4\pi\rho$$

Now replace $\mathbf{A}$ with $\mathbf{A}+\nabla\lambda$ and replace u with $u-(1/c)\partial\lambda/\partial t$ in the second result of part (b).

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta\mathbf{A} + \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial u}{\partial t}\right) = \frac{4\pi}{c}\mathbf{J}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\mathbf{A}+\nabla\lambda) - \Delta(\mathbf{A}+\nabla\lambda) + \nabla\left[\nabla\cdot(\mathbf{A}+\nabla\lambda) + \frac{1}{c}\frac{\partial}{\partial t}\left(u - \frac{1}{c}\frac{\partial\lambda}{\partial t}\right)\right] = \frac{4\pi}{c}\mathbf{J}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{1}{c^2}\nabla\left(\frac{\partial^2\lambda}{\partial t^2}\right) - \Delta\mathbf{A} - \Delta(\nabla\lambda) + \nabla\left(\nabla\cdot\mathbf{A} + \nabla\cdot\nabla\lambda + \frac{1}{c}\frac{\partial u}{\partial t} - \frac{1}{c^2}\frac{\partial^2\lambda}{\partial t^2}\right) = \frac{4\pi}{c}\mathbf{J}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \cancel{\frac{1}{c^2}\nabla\left(\frac{\partial^2\lambda}{\partial t^2}\right)} - \Delta\mathbf{A} - \cancel{\Delta(\nabla\lambda) + \nabla(\Delta\lambda)} - \cancel{\frac{1}{c^2}\nabla\left(\frac{\partial^2\lambda}{\partial t^2}\right)} + \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial u}{\partial t}\right) = \frac{4\pi}{c}\mathbf{J}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta\mathbf{A} + \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial u}{\partial t}\right) = \frac{4\pi}{c}\mathbf{J}$$

Part (d)

In order to complete the definition of $\mathbf{A}$, we require that its divergence satisfies

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial u}{\partial t} = 0$$

to simplify the second result of part (b). This is the infamous Lorenz gauge. Let $\mathbf{A}'$ represent the new $\mathbf{A}$, and let u' represent the new u.

$$\mathbf{A} = \mathbf{A}' + \nabla\lambda$$
$$u = u' - \frac{1}{c}\frac{\partial \lambda}{\partial t}$$

Take the divergence of both sides of the first equation, and differentiate both sides of the second equation with respect to t.

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}' + \nabla \cdot \nabla\lambda$$
$$\frac{\partial u}{\partial t} = \frac{\partial u'}{\partial t} - \frac{1}{c}\frac{\partial^2 \lambda}{\partial t^2}$$

Divide both sides of the second equation by c.

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}' + \Delta\lambda$$
$$\frac{1}{c}\frac{\partial u}{\partial t} = \frac{1}{c}\frac{\partial u'}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2}$$

Add the respective sides of each equation.

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial u}{\partial t} = \left(\nabla \cdot \mathbf{A}' + \frac{1}{c}\frac{\partial u'}{\partial t}\right) + \Delta\lambda - \frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2}$$

Solve for the primed variables in parentheses.

$$\nabla \cdot \mathbf{A}' + \frac{1}{c}\frac{\partial u'}{\partial t} = -\Delta\lambda + \frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2} + \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial u}{\partial t}$$

Choose λ so that the right side is zero.

$$-\Delta\lambda + \frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2} + \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial u}{\partial t} = 0$$

For the Lorenz gauge to hold then, λ must be related to the old variables by

$$\Delta\lambda - \frac{1}{c^2}\frac{\partial^2 \lambda}{\partial t^2} = \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial u}{\partial t}.$$

Part (e)

With the Lorenz gauge, the second result of part (b) reduces to

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} + \nabla\Bigg(\underbrace{\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial u}{\partial t}}_{=\,0}\Bigg) = \frac{4\pi}{c}\mathbf{J} \quad \rightarrow \quad \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} = \frac{4\pi}{c}\mathbf{J} \quad \Rightarrow \quad \begin{cases} \frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2} - \Delta A_x = \frac{4\pi}{c}J_x \\ \frac{1}{c^2}\frac{\partial^2 A_y}{\partial t^2} - \Delta A_y = \frac{4\pi}{c}J_y \\ \frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2} - \Delta A_z = \frac{4\pi}{c}J_z \end{cases},$$

and the first result of part (b) reduces to

$$-\frac{1}{c}\nabla \cdot \frac{\partial \mathbf{A}}{\partial t} - \Delta u = 4\pi\rho \quad \rightarrow \quad -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) - \Delta u = 4\pi\rho \quad \rightarrow \quad -\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial u}{\partial t}\right) - \Delta u = 4\pi\rho$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = 4\pi\rho.$$

Therefore, by introducing the potential functions, u and $\mathbf{A}$, Maxwell's equations reduce to a system of four decoupled three-dimensional inhomogeneous wave equations. Once they're solved for, the electric and magnetic fields are obtained by

$$\mathbf{E} = -\nabla u - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}.$$