Exercise 4

Show that each component of $\mathbf{E}$ and of $\mathbf{B}$ satisfies the wave equation.

Solution

The inhomogeneous Maxwell equations are

$$\begin{aligned}
&\text{(I)} && \frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - 4\pi \mathbf{J} \\
&\text{(II)} && \frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} \\
&\text{(III)} && \nabla \cdot \mathbf{E} = 4\pi\rho \\
&\text{(IV)} && \nabla \cdot \mathbf{B} = 0.
\end{aligned}$$

Differentiate both sides of equation (I) with respect to t and then use equation (II) for $\partial \mathbf{B}/\partial t$.

$$\begin{aligned}
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= \frac{\partial}{\partial t}(c\nabla \times \mathbf{B} - 4\pi \mathbf{J}) \\
&= c\nabla \times \frac{\partial \mathbf{B}}{\partial t} - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
&= c\nabla \times (-c\nabla \times \mathbf{E}) - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
&= -c^2 \nabla \times (\nabla \times \mathbf{E}) - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
&= -c^2 \left(\sum_{i=1}^3 \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\left(\sum_{j=1}^3 \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) \times \left(\sum_{k=1}^3 \boldsymbol{\delta}_k E_k\right)\right] - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
&= -c^2 \left(\sum_{i=1}^3 \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\sum_{j=1}^3 \sum_{k=1}^3 (\boldsymbol{\delta}_j \times \boldsymbol{\delta}_k) \frac{\partial E_k}{\partial x_j}\right] - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
&= -c^2 \left(\sum_{i=1}^3 \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \boldsymbol{\delta}_l \varepsilon_{jkl} \frac{\partial E_k}{\partial x_j}\right) - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
&= -c^2 \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial E_k}{\partial x_j} - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
&= -c^2 \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{m=1}^3 \boldsymbol{\delta}_m \varepsilon_{ilm} \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial E_k}{\partial x_j} - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
&= -c^2 \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{m=1}^3 \boldsymbol{\delta}_m \varepsilon_{mil} \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial E_k}{\partial x_j} - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
&= -c^2 \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \boldsymbol{\delta}_m (\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij}) \frac{\partial}{\partial x_i} \frac{\partial E_k}{\partial x_j} - 4\pi \frac{\partial \mathbf{J}}{\partial t}
\end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mj}\delta_{ik}\frac{\partial}{\partial x_i}\frac{\partial E_k}{\partial x_j} - \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{m=1}^{3}\boldsymbol{\delta}_m\delta_{mk}\delta_{ij}\frac{\partial}{\partial x_i}\frac{\partial E_k}{\partial x_j}\right) - 4\pi\frac{\partial \mathbf{J}}{\partial t} \\
&= -c^2\left(\sum_{i=1}^{3}\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_i}\frac{\partial E_i}{\partial x_j} - \sum_{i=1}^{3}\sum_{k=1}^{3}\boldsymbol{\delta}_k\frac{\partial}{\partial x_i}\frac{\partial E_k}{\partial x_i}\right) - 4\pi\frac{\partial \mathbf{J}}{\partial t} \\
&= -c^2\left[\sum_{j=1}^{3}\boldsymbol{\delta}_j\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{3}\frac{\partial E_i}{\partial x_i}\right) - \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}\left(\sum_{k=1}^{3}\boldsymbol{\delta}_k E_k\right)\right] - 4\pi\frac{\partial \mathbf{J}}{\partial t} \\
&= -c^2[\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}] - 4\pi\frac{\partial \mathbf{J}}{\partial t}
\end{aligned}$$

Substitute equation (III) here for the divergence of $\mathbf{E}$.

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = -c^2[\nabla(4\pi\rho) - \Delta\mathbf{E}] - 4\pi\frac{\partial \mathbf{J}}{\partial t}$$

Therefore, the electric field (more specifically each of its components) satisfies the inhomogeneous wave equation.

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2\Delta\mathbf{E} - 4\pi\left(c^2\nabla\rho + \frac{\partial \mathbf{J}}{\partial t}\right) \quad \Rightarrow \quad \begin{cases} \dfrac{\partial^2 E_x}{\partial t^2} = c^2\Delta E_x - 4\pi\left(c^2\dfrac{\partial\rho}{\partial x} + \dfrac{\partial J_x}{\partial t}\right) \\ \dfrac{\partial^2 E_y}{\partial t^2} = c^2\Delta E_y - 4\pi\left(c^2\dfrac{\partial\rho}{\partial y} + \dfrac{\partial J_y}{\partial t}\right) \\ \dfrac{\partial^2 E_z}{\partial t^2} = c^2\Delta E_z - 4\pi\left(c^2\dfrac{\partial\rho}{\partial z} + \dfrac{\partial J_z}{\partial t}\right) \end{cases}$$

Differentiate both sides of equation (II) with respect to t and then use equation (I) for $\partial\mathbf{E}/\partial t$.

$$\begin{aligned}
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= \frac{\partial}{\partial t}(-c\nabla\times\mathbf{E}) \\
&= -c\nabla\times\frac{\partial \mathbf{E}}{\partial t} \\
&= -c\nabla\times(c\nabla\times\mathbf{B} - 4\pi\mathbf{J}) \\
&= -c^2\nabla\times(\nabla\times\mathbf{B}) + 4\pi c(\nabla\times\mathbf{J}) \\
&= -c^2[\nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B}] + 4\pi c(\nabla\times\mathbf{J})
\end{aligned}$$

Substitute equation (IV) here for the divergence of $\mathbf{B}$.

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = -c^2[\nabla(0) - \Delta\mathbf{B}] + 4\pi c\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ J_x & J_y & J_z \end{vmatrix}$$

Therefore, the magnetic field (more specifically each of its components) satisfies the inhomogeneous wave equation.

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \Delta \mathbf{B} + 4\pi c (\nabla \times \mathbf{J}) \quad \Rightarrow \quad \begin{cases} \dfrac{\partial^2 B_x}{\partial t^2} = c^2 \Delta B_x + 4\pi c \left(\dfrac{\partial J_z}{\partial y} - \dfrac{\partial J_y}{\partial z} \right) \\ \dfrac{\partial^2 B_y}{\partial t^2} = c^2 \Delta B_y + 4\pi c \left(\dfrac{\partial J_x}{\partial z} - \dfrac{\partial J_z}{\partial x} \right) \\ \dfrac{\partial^2 B_z}{\partial t^2} = c^2 \Delta B_z + 4\pi c \left(\dfrac{\partial J_y}{\partial x} - \dfrac{\partial J_x}{\partial y} \right) \end{cases}$$