

Exercise 5

Derive carefully the formulas (8) and (9) for the solution of Maxwell's equations.

Solution

From the homogeneous Maxwell equations,

$$(I) \quad \frac{\partial \mathbf{E}}{\partial t} = c \nabla \times \mathbf{B}$$

$$(II) \quad \frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}$$

$$(III) \quad \nabla \cdot \mathbf{E} = 0$$

$$(IV) \quad \nabla \cdot \mathbf{B} = 0,$$

it follows that the electric and magnetic fields satisfy the homogeneous three-dimensional wave equation.

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{E}}{\partial t} \right) &= \frac{\partial}{\partial t} (c \nabla \times \mathbf{B}) & \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{B}}{\partial t} \right) &= \frac{\partial}{\partial t} (-c \nabla \times \mathbf{E}) \\ \frac{\partial^2 \mathbf{E}}{\partial t^2} &= c \nabla \times \frac{\partial \mathbf{B}}{\partial t} & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c \nabla \times \frac{\partial \mathbf{E}}{\partial t} \\ \frac{\partial^2 \mathbf{E}}{\partial t^2} &= c \nabla \times (-c \nabla \times \mathbf{E}) & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c \nabla \times (c \nabla \times \mathbf{B}) \\ \frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2 \nabla \times (\nabla \times \mathbf{E}) & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2 \nabla \times (\nabla \times \mathbf{B}) \\ \frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2 [\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}] & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2 [\nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}] \\ \frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2 [\nabla(0) - \Delta \mathbf{E}] & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2 [\nabla(0) - \Delta \mathbf{B}] \\ \frac{\partial^2 \mathbf{E}}{\partial t^2} &= c^2 \Delta \mathbf{E} & \frac{\partial^2 \mathbf{B}}{\partial t^2} &= c^2 \Delta \mathbf{B} \end{aligned}$$

The aim here is to solve them in all of space, each with two prescribed initial conditions.

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \Delta \mathbf{E}, \quad -\infty < x, y, z < \infty, t > 0 \quad \frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \Delta \mathbf{B}, \quad -\infty < x, y, z < \infty, t > 0$$

$$\mathbf{E}(x, y, z, 0) = \mathbf{E}^0(x, y, z)$$

$$\mathbf{B}(x, y, z, 0) = \mathbf{B}^0(x, y, z)$$

$$\frac{\partial \mathbf{E}}{\partial t}(x, y, z, 0) = c \nabla \times \mathbf{B}^0(x, y, z)$$

$$\frac{\partial \mathbf{B}}{\partial t}(x, y, z, 0) = -c \nabla \times \mathbf{E}^0(x, y, z)$$

Their solutions are given by the formula of Kirchhoff and Poisson.

$$\mathbf{E}(x, y, z, t) = \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2 t} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \mathbf{E}^0(x_0, y_0, z_0) dS_0 \right] + \frac{1}{4\pi c^2 t} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} [c\nabla \times \mathbf{B}^0(x_0, y_0, z_0)] dS_0$$

$$\mathbf{B}(x, y, z, t) = \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2 t} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \mathbf{B}^0(x_0, y_0, z_0) dS_0 \right] + \frac{1}{4\pi c^2 t} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} [-c\nabla \times \mathbf{E}^0(x_0, y_0, z_0)] dS_0$$

Write the surface integrals in square brackets explicitly by using spherical coordinates (r_0, ϕ_0, θ_0) , where θ_0 is the angle from the polar axis.

$$\begin{aligned} x_0 - x &= ct \sin \theta_0 \cos \phi_0 \\ y_0 - y &= ct \sin \theta_0 \sin \phi_0 \\ z_0 - z &= ct \cos \theta_0 \end{aligned}$$

As a result,

$$\mathbf{E}(x, y, z, t) = \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2 t} \int_0^\pi \int_0^{2\pi} \mathbf{E}^0(ct, \phi_0, \theta_0) (c^2 t^2 \sin \theta_0 d\phi_0 d\theta_0) \right] + \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{B}^0(x_0, y_0, z_0) dS_0$$

$$\mathbf{B}(x, y, z, t) = \frac{\partial}{\partial t} \left[\frac{1}{4\pi c^2 t} \int_0^\pi \int_0^{2\pi} \mathbf{B}^0(ct, \phi_0, \theta_0) (c^2 t^2 \sin \theta_0 d\phi_0 d\theta_0) \right] - \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{E}^0(x_0, y_0, z_0) dS_0.$$

Bring the constants out in front.

$$\mathbf{E}(x, y, z, t) = \frac{\partial}{\partial t} \left[\frac{t}{4\pi} \int_0^\pi \int_0^{2\pi} \mathbf{E}^0(ct, \phi_0, \theta_0) \sin \theta_0 d\phi_0 d\theta_0 \right] + \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{B}^0(x_0, y_0, z_0) dS_0$$

$$\mathbf{B}(x, y, z, t) = \frac{\partial}{\partial t} \left[\frac{t}{4\pi} \int_0^\pi \int_0^{2\pi} \mathbf{B}^0(ct, \phi_0, \theta_0) \sin \theta_0 d\phi_0 d\theta_0 \right] - \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{E}^0(x_0, y_0, z_0) dS_0$$

Evaluate the derivative by using the product and chain rules.

$$\begin{aligned} \mathbf{E}(x, y, z, t) = & \left[\frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \mathbf{E}^0(ct, \phi_0, \theta_0) \sin \theta_0 d\phi_0 d\theta_0 + \frac{t}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\partial \mathbf{E}^0}{\partial r_0}(ct, \phi_0, \theta_0) \frac{\partial}{\partial t}(ct) \sin \theta_0 d\phi_0 d\theta_0 \right] \\ & + \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{B}^0(x_0, y_0, z_0) dS_0 \end{aligned}$$

$$\begin{aligned} \mathbf{B}(x, y, z, t) = & \left[\frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \mathbf{B}^0(ct, \phi_0, \theta_0) \sin \theta_0 d\phi_0 d\theta_0 + \frac{t}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\partial \mathbf{B}^0}{\partial r_0}(ct, \phi_0, \theta_0) \frac{\partial}{\partial t}(ct) \sin \theta_0 d\phi_0 d\theta_0 \right] \\ & - \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{E}^0(x_0, y_0, z_0) dS_0 \end{aligned}$$

Evaluate the remaining derivative.

$$\mathbf{E}(x, y, z, t) = \left[\frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \mathbf{E}^0(ct, \phi_0, \theta_0) \sin \theta_0 d\phi_0 d\theta_0 + \frac{t}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\partial \mathbf{E}^0}{\partial r_0}(ct, \phi_0, \theta_0)(c) \sin \theta_0 d\phi_0 d\theta_0 \right] \\ + \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{B}^0(x_0, y_0, z_0) dS_0$$

$$\mathbf{B}(x, y, z, t) = \left[\frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} \mathbf{B}^0(ct, \phi_0, \theta_0) \sin \theta_0 d\phi_0 d\theta_0 + \frac{t}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\partial \mathbf{B}^0}{\partial r_0}(ct, \phi_0, \theta_0)(c) \sin \theta_0 d\phi_0 d\theta_0 \right] \\ - \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{E}^0(x_0, y_0, z_0) dS_0$$

Factor $1/(4\pi ct)$ from the square brackets and prepare to write the differential as dS_0 again.

$$\mathbf{E}(x, y, z, t) = \frac{1}{4\pi ct} \left[\frac{1}{ct} \int_0^\pi \int_0^{2\pi} \mathbf{E}^0(ct, \phi_0, \theta_0)(c^2t^2 \sin \theta_0 d\phi_0 d\theta_0) + \int_0^\pi \int_0^{2\pi} \frac{\partial \mathbf{E}^0}{\partial r_0}(ct, \phi_0, \theta_0)(c^2t^2 \sin \theta_0 d\phi_0 d\theta_0) \right] \\ + \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{B}^0(x_0, y_0, z_0) dS_0$$

$$\mathbf{B}(x, y, z, t) = \frac{1}{4\pi ct} \left[\frac{1}{ct} \int_0^\pi \int_0^{2\pi} \mathbf{B}^0(ct, \phi_0, \theta_0)(c^2t^2 \sin \theta_0 d\phi_0 d\theta_0) + \int_0^\pi \int_0^{2\pi} \frac{\partial \mathbf{B}^0}{\partial r_0}(ct, \phi_0, \theta_0)(c^2t^2 \sin \theta_0 d\phi_0 d\theta_0) \right] \\ - \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{E}^0(x_0, y_0, z_0) dS_0$$

Change the integrals in square brackets back to Cartesian coordinates.

$$\mathbf{E}(x, y, z, t) = \frac{1}{4\pi ct} \left[\frac{1}{ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \mathbf{E}^0(x_0, y_0, z_0) dS_0 + \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \frac{\partial \mathbf{E}^0}{\partial r_0}(x_0, y_0, z_0) dS_0 \right] + \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{B}^0(x_0, y_0, z_0) dS_0$$

$$\mathbf{B}(x, y, z, t) = \frac{1}{4\pi ct} \left[\frac{1}{ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \mathbf{B}^0(x_0, y_0, z_0) dS_0 + \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \frac{\partial \mathbf{B}^0}{\partial r_0}(x_0, y_0, z_0) dS_0 \right] - \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \nabla \times \mathbf{E}^0(x_0, y_0, z_0) dS_0$$

Therefore, combining the integrals, equations (8) and (9) for the electric and magnetic fields are obtained.

$$\mathbf{E}(x, y, z, t) = \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \left[\frac{1}{ct} \mathbf{E}^0(x_0, y_0, z_0) + \frac{\partial \mathbf{E}^0}{\partial r_0}(x_0, y_0, z_0) + \nabla \times \mathbf{B}^0(x_0, y_0, z_0) \right] dS_0$$

$$\mathbf{B}(x, y, z, t) = \frac{1}{4\pi ct} \iint_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \left[\frac{1}{ct} \mathbf{B}^0(x_0, y_0, z_0) + \frac{\partial \mathbf{B}^0}{\partial r_0}(x_0, y_0, z_0) - \nabla \times \mathbf{E}^0(x_0, y_0, z_0) \right] dS_0$$