Exercise 8

Solve the inhomogeneous Maxwell equations.

Solution

From the inhomogeneous Maxwell equations,

$$\begin{aligned}
&\text{(I)} \quad \frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - 4\pi \mathbf{J} \\
&\text{(II)} \quad \frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} \\
&\text{(III)} \quad \nabla \cdot \mathbf{E} = 4\pi\rho \\
&\text{(IV)} \quad \nabla \cdot \mathbf{B} = 0,
\end{aligned}$$

it follows that the electric and magnetic fields satisfy the inhomogeneous wave equation.

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t}\right) &= \frac{\partial}{\partial t}(c\nabla \times \mathbf{B} - 4\pi \mathbf{J}) \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= c\nabla \times \frac{\partial \mathbf{B}}{\partial t} - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= c\nabla \times (-c\nabla \times \mathbf{E}) - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2\nabla \times (\nabla \times \mathbf{E}) - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2[\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}] - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= -c^2[\nabla(4\pi\rho) - \Delta \mathbf{E}] - 4\pi \frac{\partial \mathbf{J}}{\partial t} \\
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= c^2\Delta \mathbf{E} - 4\pi \frac{\partial \mathbf{J}}{\partial t} - 4\pi c^2 \nabla\rho
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{B}}{\partial t}\right) &= \frac{\partial}{\partial t}(-c\nabla \times \mathbf{E}) \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c\nabla \times \frac{\partial \mathbf{E}}{\partial t} \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c\nabla \times (c\nabla \times \mathbf{B} - 4\pi \mathbf{J}) \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2\nabla \times (\nabla \times \mathbf{B}) + 4\pi c\nabla \times \mathbf{J} \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2[\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}] + 4\pi c\nabla \times \mathbf{J} \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= -c^2[\nabla(0) - \Delta \mathbf{B}] + 4\pi c\nabla \times \mathbf{J} \\
\frac{\partial^2 \mathbf{B}}{\partial t^2} &= c^2\Delta \mathbf{B} + 4\pi c\nabla \times \mathbf{J}
\end{aligned}$$

The aim here is to solve them in all of space ($-\infty < x, y, z < \infty$) for $t > 0$, each with zero initial conditions.

$$\mathbf{E}(x, y, z, 0) = \mathbf{0} \qquad \mathbf{B}(x, y, z, 0) = \mathbf{0}$$

$$\frac{\partial \mathbf{E}}{\partial t}(x, y, z, 0) = \mathbf{0} \qquad \frac{\partial \mathbf{B}}{\partial t}(x, y, z, 0) = \mathbf{0}$$

According to Duhamel's principle, the solutions to these inhomogeneous wave equations are

$$\mathbf{E}(x, y, z, t) = \int_0^t \mathbf{E}_h(x, y, z, t - s; s)\, ds \quad \text{and} \quad \mathbf{B}(x, y, z, t) = \int_0^t \mathbf{B}_h(x, y, z, t - s; s)\, ds,$$

where $\mathbf{E}_h(x, y, z, t; s)$ and $\mathbf{B}_h(x, y, z, t; s)$ are the respective solutions to the associated homogeneous equations with a particular choice for the initial conditions.

$$\frac{\partial^2 \mathbf{E}_h}{\partial t^2} = c^2 \Delta \mathbf{E}_h, \quad -\infty < x, y, z < \infty, \ t > 0 \qquad \frac{\partial^2 \mathbf{B}_h}{\partial t^2} = c^2 \Delta \mathbf{B}_h, \quad -\infty < x, y, z < \infty, \ t > 0$$

$$\mathbf{E}_h(x, y, z, 0; s) = \mathbf{0} \qquad \mathbf{B}_h(x, y, z, 0; s) = \mathbf{0}$$

$$\frac{\partial \mathbf{E}_h}{\partial t}(x, y, z, 0; s) = -4\pi \frac{\partial \mathbf{J}}{\partial t}(x, y, z, s) - 4\pi c^2 (\nabla \rho)(x, y, z, s) \qquad \frac{\partial \mathbf{B}_h}{\partial t}(x, y, z, 0; s) = 4\pi c (\nabla \times \mathbf{J})(x, y, z, s)$$

The solutions for $\mathbf{E}_h$ and $\mathbf{B}_h$ are given by the formula of Kirchhoff and Poisson (Section 9.2).

$$\begin{aligned}
\mathbf{E}_h(x, y, z, t; s) &= \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \left[-4\pi \frac{\partial \mathbf{J}}{\partial t}(x_0, y_0, z_0, s) - 4\pi c^2 (\nabla_0 \rho)(x_0, y_0, z_0, s) \right] dS_0 \\
&= -\frac{1}{c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \left[\frac{\partial \mathbf{J}}{\partial t}(x_0, y_0, z_0, s) + c^2 (\nabla_0 \rho)(x_0, y_0, z_0, s) \right] dS_0
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}_h(x, y, z, t; s) &= \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} [4\pi c (\nabla_0 \times \mathbf{J})(x_0, y_0, z_0, s)] \, dS_0 \\
&= \frac{1}{ct} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} (\nabla_0 \times \mathbf{J})(x_0, y_0, z_0, s) \, dS_0
\end{aligned}$$

Now $\mathbf{E}$ and $\mathbf{B}$ can be determined.

$$\begin{aligned}
\mathbf{E}(x, y, z, t) &= \int_0^t \mathbf{E}_h(x, y, z, t - s; s) \, ds \\
&= \int_0^t \left\{ -\frac{1}{c^2 (t - s)} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2(t-s)^2}} \left[\frac{\partial \mathbf{J}}{\partial t}(x_0, y_0, z_0, s) + c^2 (\nabla_0 \rho)(x_0, y_0, z_0, s) \right] dS_0 \right\} ds \\
&= -\frac{1}{c^2} \int_0^t \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2(t-s)^2}} \frac{1}{t - s} \left[\frac{\partial \mathbf{J}}{\partial t}(x_0, y_0, z_0, s) + c^2 (\nabla_0 \rho)(x_0, y_0, z_0, s) \right] dS_0 \, ds
\end{aligned}$$

Write the surface integral over the sphere centered at (x, y, z) with radius $c(t - s)$ explicitly by using spherical coordinates (r_0, θ_0, ϕ_0), where ϕ_0 represents the angle from the polar axis.

$$\begin{aligned}
x_0 - x &= c(t - s) \sin \phi_0 \cos \theta_0 \\
y_0 - y &= c(t - s) \sin \phi_0 \sin \theta_0 \\
z_0 - z &= c(t - s) \cos \phi_0
\end{aligned}$$

The solution becomes

$$\begin{aligned}\mathbf{E}(x,y,z,t) &= -\frac{1}{c^2}\int_0^t\int_0^\pi\int_0^{2\pi}\frac{1}{t-s}\left[\frac{\partial\mathbf{J}}{\partial t}(c(t-s),\theta_0,\phi_0,s)+c^2(\nabla_0\rho)(c(t-s),\theta_0,\phi_0,s)\right]c^2(t-s)^2\sin\phi_0\,d\theta_0\,d\phi_0\,ds\\ &= \frac{1}{c^2}\int_0^t\int_0^\pi\int_0^{2\pi}\left[\frac{\partial\mathbf{J}}{\partial t}(c(t-s),\theta_0,\phi_0,s)+c^2(\nabla_0\rho)(c(t-s),\theta_0,\phi_0,s)\right]c(t-s)\sin\phi_0\,d\theta_0\,d\phi_0\,(-c\,ds).\end{aligned}$$

Make the following substitution.

$$\begin{aligned}r_0 &= c(t-s) \quad\to\quad s = t-\frac{r_0}{c}\\ dr_0 &= -c\,ds\end{aligned}$$

As a result,

$$\begin{aligned}\mathbf{E}(x,y,z,t) &= \frac{1}{c^2}\int_{ct}^0\int_0^\pi\int_0^{2\pi}\left[\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)+c^2(\nabla_0\rho)\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right]r_0\sin\phi_0\,d\theta_0\,d\phi_0\,dr_0\\ &= \int_0^\pi\int_0^{2\pi}\int_0^{ct}\left[-\frac{1}{c^2}\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)-(\nabla_0\rho)\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right]r_0\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0.\end{aligned}$$

Consider the gradient in $x_0y_0z_0$-space of $\rho(r_0,\theta_0,\phi_0,t-r_0/c)$.

$$\begin{aligned}\nabla_0\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) &= \left(\hat{\mathbf{r}}_0\frac{\partial}{\partial r_0}+\frac{\hat{\boldsymbol{\phi}}_0}{r_0}\frac{\partial}{\partial\phi_0}+\frac{\hat{\boldsymbol{\theta}}_0}{r_0\sin\phi_0}\frac{\partial}{\partial\theta_0}\right)\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\\ &= \hat{\mathbf{r}}_0\left[\frac{\partial\rho}{\partial r_0}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)+\frac{\partial\rho}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\frac{\partial}{\partial r_0}\left(t-\frac{r_0}{c}\right)\right]\\ &\quad+\frac{\hat{\boldsymbol{\phi}}_0}{r_0}\frac{\partial\rho}{\partial\phi_0}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\\ &\quad+\frac{\hat{\boldsymbol{\theta}}_0}{r_0\sin\phi_0}\frac{\partial\rho}{\partial\theta_0}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\\ &= \hat{\mathbf{r}}_0\left[\frac{\partial\rho}{\partial r_0}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)-\frac{1}{c}\frac{\partial\rho}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right]\\ &\quad+\frac{\hat{\boldsymbol{\phi}}_0}{r_0}\frac{\partial\rho}{\partial\phi_0}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\\ &\quad+\frac{\hat{\boldsymbol{\theta}}_0}{r_0\sin\phi_0}\frac{\partial\rho}{\partial\theta_0}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\\ &= \hat{\mathbf{r}}_0\frac{\partial\rho}{\partial r_0}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)+\frac{\hat{\boldsymbol{\phi}}_0}{r_0}\frac{\partial\rho}{\partial\phi_0}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\\ &\quad+\frac{\hat{\boldsymbol{\theta}}_0}{r_0\sin\phi_0}\frac{\partial\rho}{\partial\theta_0}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial\rho}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\\ &= (\nabla_0\rho)\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial\rho}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\end{aligned}$$

That means

$$(\nabla_0\rho)\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)=\left(\hat{\mathbf{r}}_0\frac{\partial}{\partial r_0}+\frac{\hat{\boldsymbol{\phi}}_0}{r_0}\frac{\partial}{\partial\phi_0}+\frac{\hat{\boldsymbol{\theta}}_0}{r_0\sin\phi_0}\frac{\partial}{\partial\theta_0}\right)\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)+\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial\rho}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right).$$

Noting the following unit vector derivatives,

$$\frac{\partial}{\partial r_0}\hat{\mathbf{r}}_0 = \mathbf{0} \quad \text{and} \quad \frac{\partial}{\partial \phi_0}\hat{\boldsymbol{\phi}}_0 = -\hat{\mathbf{r}}_0 \quad \text{and} \quad \frac{\partial}{\partial \theta_0}\hat{\boldsymbol{\theta}}_0 = -\hat{\mathbf{r}}_0 \sin\phi_0 - \hat{\boldsymbol{\phi}}_0 \cos\phi_0,$$

substitute this formula for the gradient into the integral and split the integral up.

$$\begin{aligned}
\mathbf{E}(x,y,z,t) &= \int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[-\frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) - (\nabla_0\rho)\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&= \int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial \rho}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&\quad - \int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[\hat{\mathbf{r}}_0 \frac{\partial}{\partial r_0}\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) + \frac{\hat{\boldsymbol{\phi}}_0}{r_0}\frac{\partial}{\partial \phi_0}\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) + \frac{\hat{\boldsymbol{\theta}}_0}{r_0\sin\phi_0}\frac{\partial}{\partial \theta_0}\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&= \int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial \rho}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&\quad - \int_0^\pi \int_0^{2\pi} \int_0^{ct} \hat{\mathbf{r}}_0 r_0 \frac{\partial}{\partial r_0}\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&\qquad - \int_0^\pi \int_0^{2\pi} \int_0^{ct} \hat{\boldsymbol{\phi}}_0 \sin\phi_0 \frac{\partial}{\partial \phi_0}\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) dr_0 \, d\theta_0 \, d\phi_0 \\
&\qquad\quad - \int_0^\pi \int_0^{2\pi} \int_0^{ct} \hat{\boldsymbol{\theta}}_0 \frac{\partial}{\partial \theta_0}\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) dr_0 \, d\theta_0 \, d\phi_0 \\
&= \int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial \rho}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&\quad - \int_0^\pi \int_0^{2\pi} \left[\int_0^{ct} \hat{\mathbf{r}}_0 r_0 \frac{\partial}{\partial r_0}\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) dr_0\right] \sin\phi_0 \, d\theta_0 \, d\phi_0 \\
&\qquad - \int_0^{2\pi} \int_0^{ct} \left[\int_0^\pi \hat{\boldsymbol{\phi}}_0 \sin\phi_0 \frac{\partial}{\partial \phi_0}\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) d\phi_0\right] dr_0 \, d\theta_0 \\
&\qquad\quad - \int_0^\pi \int_0^{ct} \left[\int_0^{2\pi} \hat{\boldsymbol{\theta}}_0 \frac{\partial}{\partial \theta_0}\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) d\theta_0\right] dr_0 \, d\phi_0
\end{aligned}$$

Continue the simplification by integrating by parts.

$$\begin{aligned}
\mathbf{E}(x,y,z,t) &= \int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[-\frac{\hat{\mathbf{r}}_0}{c} \frac{\partial \rho}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&\quad - \int_0^\pi \int_0^{2\pi} \left[\hat{\mathbf{r}}_0 \, r_0 \rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\bigg|_0^{ct} - \int_0^{ct} \frac{\partial}{\partial r_0}(\hat{\mathbf{r}}_0 r_0)\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) dr_0 \right] \sin\phi_0 \, d\theta_0 \, d\phi_0 \\
&\quad - \int_0^{2\pi} \int_0^{ct} \left[\hat{\boldsymbol{\phi}}_0 \sin\phi_0 \, \rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\bigg|_0^{\pi} - \int_0^{\pi} \frac{\partial}{\partial \phi_0}(\hat{\boldsymbol{\phi}}_0 \sin\phi_0)\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) d\phi_0 \right] dr_0 \, d\theta_0 \\
&\quad - \int_0^{\pi} \int_0^{ct} \left[\hat{\boldsymbol{\theta}}_0 \, \rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\bigg|_0^{2\pi} - \int_0^{2\pi} \frac{\partial}{\partial \theta_0}(\hat{\boldsymbol{\theta}}_0)\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) d\theta_0 \right] dr_0 \, d\phi_0 \\
&= \int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[-\frac{\hat{\mathbf{r}}_0}{c} \frac{\partial \rho}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&\quad - \int_0^\pi \int_0^{2\pi} \left[\hat{\mathbf{r}}_0 \, ct \, \rho(ct,\theta_0,\phi_0,0) - 0 \right] \sin\phi_0 \, d\theta_0 \, d\phi_0 \\
&\quad + \int_0^\pi \int_0^{2\pi} \int_0^{ct} \hat{\mathbf{r}}_0 \, \rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) dr_0 \, \sin\phi_0 \, d\theta_0 \, d\phi_0 \\
&\quad - \int_0^{2\pi} \int_0^{ct} \left[\hat{\boldsymbol{\phi}}_0 \, \cancel{\rho\left(r_0,\theta_0,\pi,t-\frac{r_0}{c}\right)} \sin\pi - \hat{\boldsymbol{\phi}}_0 \, \cancel{\rho\left(r_0,\theta_0,0,t-\frac{r_0}{c}\right)} \sin 0 \right] dr_0 \, d\theta_0 \\
&\quad + \int_0^{2\pi} \int_0^{ct} \int_0^{\pi} (-\hat{\mathbf{r}}_0 \sin\phi_0 + \cancel{\hat{\boldsymbol{\phi}}_0 \cos\phi_0})\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) d\phi_0 \, dr_0 \, d\theta_0 \\
&\quad - \int_0^{\pi} \int_0^{ct} \left[\hat{\boldsymbol{\theta}}_0 \, \cancel{\rho\left(r_0,2\pi,\phi_0,t-\frac{r_0}{c}\right) - \hat{\boldsymbol{\theta}}_0 \, \rho\left(r_0,0,\phi_0,t-\frac{r_0}{c}\right)} \right] dr_0 \, d\phi_0 \\
&\quad + \int_0^{\pi} \int_0^{ct} \int_0^{2\pi} (-\hat{\mathbf{r}}_0 \sin\phi_0 - \cancel{\hat{\boldsymbol{\phi}}_0 \cos\phi_0})\rho\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) d\phi_0 \, dr_0 \, d\theta_0
\end{aligned}$$

Finally, use the fact that $\nabla \cdot \mathbf{E}(x, y, z, 0) = 4\pi\rho(x, y, z, 0) = 0$.

$$\begin{aligned}
\mathbf{E}(x, y, z, t) &= \int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[-\frac{\hat{\mathbf{r}}_0}{c}\frac{\partial\rho}{\partial t}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right) - \frac{1}{c^2}\frac{\partial \mathbf{J}}{\partial t}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&\quad \cancel{- \int_0^{\pi}\int_0^{2\pi} \hat{\mathbf{r}}_0 \, ct \, \rho(ct, \theta_0, \phi_0, 0) \sin\phi_0 \, d\theta_0 \, d\phi_0} \\
&\quad - \int_0^{\pi}\int_0^{2\pi}\int_0^{ct} \hat{\mathbf{r}}_0 \, \rho\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right) dr_0 \, \sin\phi_0 \, d\theta_0 \, d\phi_0 \\
&= \int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[-\frac{\hat{\mathbf{r}}_0}{r_0^2}\rho\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right) - \frac{\hat{\mathbf{r}}_0}{cr_0}\frac{\partial\rho}{\partial t}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right) - \frac{1}{c^2 r_0}\frac{\partial \mathbf{J}}{\partial t}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right)\right] r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&= \int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[-\frac{\mathbf{r}_0}{r_0^3}\rho\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right) - \frac{\mathbf{r}_0}{cr_0^2}\frac{\partial\rho}{\partial t}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right) - \frac{1}{c^2 r_0}\frac{\partial \mathbf{J}}{\partial t}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right)\right] r_0^2 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0
\end{aligned}$$

Therefore, changing back to Cartesian coordinates,

$$\boxed{\begin{aligned}
\mathbf{E}(x, y, z, t) = \iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2 \le c^2t^2}} &\left\{\frac{(x - x_0)\hat{\mathbf{x}} + (y - y_0)\hat{\mathbf{y}} + (z - z_0)\hat{\mathbf{z}}}{\sqrt{[(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2]^3}}\,\rho\left(x_0, y_0, z_0, t - \frac{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}}{c}\right)\right. \\
&+ \frac{1}{c}\frac{(x - x_0)\hat{\mathbf{x}} + (y - y_0)\hat{\mathbf{y}} + (z - z_0)\hat{\mathbf{z}}}{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}\frac{\partial\rho}{\partial t}\left(x_0, y_0, z_0, t - \frac{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}}{c}\right) \\
&\left. - \frac{1}{c^2}\frac{1}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}}\frac{\partial \mathbf{J}}{\partial t}\left(x_0, y_0, z_0, t - \frac{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}}{c}\right)\right\} dV_0.
\end{aligned}}$$

Turning our attention back to the magnetic field, plug the formula for $\mathbf{B}_h$ into the one for $\mathbf{B}$.

$$\begin{aligned}
\mathbf{B}(x,y,z,t) &= \int_0^t \mathbf{B}_h(x,y,z,t-s;s)\,ds \\
&= \int_0^t \left\{ \frac{1}{c(t-s)} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2(t-s)^2}} (\nabla_0 \times \mathbf{J})(x_0,y_0,z_0,s)\,dS_0 \right\} ds \\
&= \frac{1}{c}\int_0^t \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2(t-s)^2}} \frac{1}{t-s}\left[(\nabla_0 \times \mathbf{J})(x_0,y_0,z_0,s)\right] dS_0\,ds
\end{aligned}$$

Write the surface integral over the sphere centered at (x,y,z) with radius $c(t-s)$ explicitly by using spherical coordinates (r_0,θ_0,ϕ_0), where ϕ_0 represents the angle from the polar axis.

$$\begin{aligned}
x_0 - x &= c(t-s)\sin\theta_0\cos\phi_0 \\
y_0 - y &= c(t-s)\sin\theta_0\sin\phi_0 \\
z_0 - z &= c(t-s)\cos\theta_0
\end{aligned}$$

The solution becomes

$$\begin{aligned}
\mathbf{B}(x,y,z,t) &= \frac{1}{c}\int_0^t\int_0^\pi\int_0^{2\pi} \frac{1}{t-s}\left[(\nabla_0\times\mathbf{J})(c(t-s),\theta_0,\phi_0,s)\right] c^2(t-s)^2\sin\phi_0\,d\theta_0\,d\phi_0\,ds \\
&= -\frac{1}{c}\int_0^t\int_0^\pi\int_0^{2\pi} \left[(\nabla_0\times\mathbf{J})(c(t-s),\theta_0,\phi_0,s)\right] c(t-s)\sin\phi_0\,d\theta_0\,d\phi_0\,(-c\,ds).
\end{aligned}$$

Make the following substitution.

$$\begin{aligned}
r_0 &= c(t-s) \quad \rightarrow \quad s = t - \frac{r_0}{c} \\
dr_0 &= -c\,ds
\end{aligned}$$

As a result,

$$\begin{aligned}
\mathbf{B}(x,y,z,t) &= -\frac{1}{c}\int_{ct}^0\int_0^\pi\int_0^{2\pi} \left[(\nabla_0\times\mathbf{J})\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0\sin\phi_0\,d\theta_0\,d\phi_0\,dr_0 \\
&= \frac{1}{c}\int_0^\pi\int_0^{2\pi}\int_0^{ct} \left[(\nabla_0\times\mathbf{J})\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0.
\end{aligned}$$

Consider the curl in $x_0y_0z_0$-space of $\mathbf{J}(r_0,\theta_0,\phi_0,t-r_0/c)$.

$$\begin{aligned}
\left(\hat{\mathbf{r}}_0\frac{\partial}{\partial r_0} + \frac{\hat{\boldsymbol{\phi}}_0}{r_0}\frac{\partial}{\partial\phi_0} + \frac{\hat{\boldsymbol{\theta}}_0}{r_0\sin\phi_0}\frac{\partial}{\partial\theta_0}\right)\times\mathbf{J}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) &= \left[\frac{1}{r_0\sin\phi_0}\frac{\partial}{\partial\phi_0}(J_\theta\sin\phi_0) - \frac{1}{r_0\sin\phi_0}\frac{\partial J_\phi}{\partial\theta_0}\right]\hat{\mathbf{r}}_0 \\
&\quad + \left[\frac{1}{r_0\sin\phi_0}\frac{\partial J_r}{\partial\theta_0} - \frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0J_\theta)\right]\hat{\boldsymbol{\phi}}_0 \\
&\quad + \left[\frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0J_\phi) - \frac{1}{r_0}\frac{\partial J_r}{\partial\phi_0}\right]\hat{\boldsymbol{\theta}}_0
\end{aligned}$$

The angular derivatives are unaffected, but the radial derivative changes.

$$\begin{aligned}
\left(\hat{\mathbf{r}}_0\frac{\partial}{\partial r_0}+\frac{\hat{\boldsymbol{\phi}}_0}{r_0}\frac{\partial}{\partial\phi_0}+\frac{\hat{\boldsymbol{\theta}}_0}{r_0\sin\phi_0}\frac{\partial}{\partial\theta_0}\right)\times\mathbf{J}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) &= \left[\frac{1}{r_0\sin\phi_0}\frac{\partial}{\partial\phi_0}(J_\theta\sin\phi_0)-\frac{1}{r_0\sin\phi_0}\frac{\partial J_\phi}{\partial\theta_0}\right]\hat{\mathbf{r}}_0 \\
&\quad+\left\{\frac{1}{r_0\sin\phi_0}\frac{\partial J_r}{\partial\theta_0}-\frac{1}{r_0}\left[J_\theta+r_0\left[\frac{\partial J_\theta}{\partial r_0}+\frac{\partial J_\theta}{\partial t}\frac{\partial}{\partial r_0}\left(t-\frac{r_0}{c}\right)\right]\right]\right\}\hat{\boldsymbol{\phi}}_0 \\
&\quad+\left\{\frac{1}{r_0}\left[J_\phi+r_0\left[\frac{\partial J_\phi}{\partial r_0}+\frac{\partial J_\phi}{\partial t}\frac{\partial}{\partial r_0}\left(t-\frac{r_0}{c}\right)\right]\right]-\frac{1}{r_0}\frac{\partial J_r}{\partial\phi_0}\right\}\hat{\boldsymbol{\theta}}_0 \\
&= \left[\frac{1}{r_0\sin\phi_0}\frac{\partial}{\partial\phi_0}(J_\theta\sin\phi_0)-\frac{1}{r_0\sin\phi_0}\frac{\partial J_\phi}{\partial\theta_0}\right]\hat{\mathbf{r}}_0 \\
&\quad+\left\{\frac{1}{r_0\sin\phi_0}\frac{\partial J_r}{\partial\theta_0}-\frac{1}{r_0}\left[J_\theta+r_0\left(\frac{\partial J_\theta}{\partial r_0}-\frac{1}{c}\frac{\partial J_\theta}{\partial t}\right)\right]\right\}\hat{\boldsymbol{\phi}}_0 \\
&\quad+\left\{\frac{1}{r_0}\left[J_\phi+r_0\left(\frac{\partial J_\phi}{\partial r_0}-\frac{1}{c}\frac{\partial J_\phi}{\partial t}\right)\right]-\frac{1}{r_0}\frac{\partial J_r}{\partial\phi_0}\right\}\hat{\boldsymbol{\theta}}_0 \\
&= \left[\frac{1}{r_0\sin\phi_0}\frac{\partial}{\partial\phi_0}(J_\theta\sin\phi_0)-\frac{1}{r_0\sin\phi_0}\frac{\partial J_\phi}{\partial\theta_0}\right]\hat{\mathbf{r}}_0 \\
&\quad+\left\{\frac{1}{r_0\sin\phi_0}\frac{\partial J_r}{\partial\theta_0}-\frac{1}{r_0}\left(J_\theta+r_0\frac{\partial J_\theta}{\partial r_0}\right)\right\}\hat{\boldsymbol{\phi}}_0+\frac{1}{c}\frac{\partial J_\theta}{\partial t}\hat{\boldsymbol{\phi}}_0 \\
&\quad+\left\{\frac{1}{r_0}\left(J_\phi+r_0\frac{\partial J_\phi}{\partial r_0}\right)-\frac{1}{r_0}\frac{\partial J_r}{\partial\phi_0}\right\}\hat{\boldsymbol{\theta}}_0-\frac{1}{c}\frac{\partial J_\phi}{\partial t}\hat{\boldsymbol{\theta}}_0 \\
&= (\nabla_0\times\mathbf{J})\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)+\frac{1}{c}\left[\frac{\partial J_\theta}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\hat{\boldsymbol{\phi}}_0-\frac{\partial J_\phi}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\hat{\boldsymbol{\theta}}_0\right] \\
&= (\nabla_0\times\mathbf{J})\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)+\frac{1}{c}\left[-\frac{\partial J_r}{\partial t}(\hat{\mathbf{r}}_0\times\hat{\mathbf{r}}_0)+\frac{\partial J_\theta}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)(-\hat{\mathbf{r}}_0\times\hat{\boldsymbol{\theta}}_0)\right. \\
&\qquad\left.-\frac{\partial J_\phi}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)(\hat{\mathbf{r}}_0\times\hat{\boldsymbol{\phi}}_0)\right] \\
&= (\nabla_0\times\mathbf{J})\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)-\frac{1}{c}\hat{\mathbf{r}}_0\times\frac{\partial\mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)
\end{aligned}$$

That means

$$(\nabla_0 \times \mathbf{J})\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right) = \left(\hat{\mathbf{r}}_0 \frac{\partial}{\partial r_0} + \frac{\hat{\boldsymbol{\phi}}_0}{r_0}\frac{\partial}{\partial \phi_0} + \frac{\hat{\boldsymbol{\theta}}_0}{r_0 \sin\phi_0}\frac{\partial}{\partial \theta_0}\right) \times \mathbf{J}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right) + \frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right).$$

Substitute this formula for the curl into the integral and split the integral up.

$$\begin{aligned}
\mathbf{B}(x,y,z,t) &= \frac{1}{c}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[(\nabla_0 \times \mathbf{J})\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&= \frac{1}{c}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&\quad + \frac{1}{c}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[\left(\hat{\mathbf{r}}_0 \frac{\partial}{\partial r_0} + \frac{\hat{\boldsymbol{\phi}}_0}{r_0}\frac{\partial}{\partial \phi_0} + \frac{\hat{\boldsymbol{\theta}}_0}{r_0 \sin\phi_0}\frac{\partial}{\partial \theta_0}\right) \times \mathbf{J}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&= \frac{1}{c}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&\quad + \frac{1}{c}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left\{\hat{\mathbf{r}}_0 \left[\frac{1}{r_0 \sin\phi_0}\frac{\partial}{\partial \phi_0}(J_\theta \sin\phi_0) - \frac{1}{r_0 \sin\phi_0}\frac{\partial}{\partial \theta_0} J_\phi\right]\right. \\
&\qquad + \hat{\boldsymbol{\phi}}_0 \left[\frac{1}{r_0 \sin\phi_0}\frac{\partial}{\partial \theta_0} J_r - \frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0 J_\theta)\right] \\
&\qquad \left. + \hat{\boldsymbol{\theta}}_0 \left[\frac{1}{r_0}\frac{\partial}{\partial r_0}(r_0 J_\phi) - \frac{1}{r_0}\frac{\partial}{\partial \phi_0} J_r\right]\right\} r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&= \frac{1}{c}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&\quad + \frac{1}{c}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left\{\hat{\mathbf{r}}_0 \left[\frac{\partial}{\partial \phi_0}(J_\theta \sin\phi_0) - \frac{\partial}{\partial \theta_0} J_\phi\right]\right. \\
&\qquad + \hat{\boldsymbol{\phi}}_0 \left[\frac{\partial}{\partial \theta_0} J_r - \sin\phi_0 \frac{\partial}{\partial r_0}(r_0 J_\theta)\right] \\
&\qquad \left. + \hat{\boldsymbol{\theta}}_0 \left[\sin\phi_0 \frac{\partial}{\partial r_0}(r_0 J_\phi) - \sin\phi_0 \frac{\partial}{\partial \phi_0} J_r\right]\right\} dr_0 \, d\theta_0 \, d\phi_0 \\
&= \frac{1}{c}\int_0^\pi \int_0^{2\pi} \int_0^{ct} \left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0, \theta_0, \phi_0, t - \frac{r_0}{c}\right)\right] r_0 \sin\phi_0 \, dr_0 \, d\theta_0 \, d\phi_0 \\
&\quad + \frac{1}{c}\int_0^{2\pi} \int_0^{ct} \left[\int_0^\pi \hat{\mathbf{r}}_0 \frac{\partial}{\partial \phi_0}(J_\theta \sin\phi_0) \, d\phi_0\right] dr_0 \, d\theta_0 - \frac{1}{c}\int_0^\pi \int_0^{ct} \left[\int_0^{2\pi} \hat{\mathbf{r}}_0 \frac{\partial}{\partial \theta_0} J_\phi \, d\theta_0\right] dr_0 \, d\phi_0 \\
&\quad + \frac{1}{c}\int_0^\pi \int_0^{ct} \left[\int_0^{2\pi} \hat{\boldsymbol{\phi}}_0 \frac{\partial}{\partial \theta_0} J_r \, d\theta_0\right] dr_0 \, d\phi_0 - \frac{1}{c}\int_0^\pi \int_0^{2\pi} \left[\int_0^{ct} \hat{\boldsymbol{\phi}}_0 \frac{\partial}{\partial r_0}(r_0 J_\theta) \, dr_0\right] \sin\phi_0 \, d\theta_0 \, d\phi_0 \\
&\quad + \frac{1}{c}\int_0^\pi \int_0^{2\pi} \left[\int_0^{ct} \hat{\boldsymbol{\theta}}_0 \frac{\partial}{\partial r_0}(r_0 J_\phi) \, dr_0\right] \sin\phi_0 \, d\theta_0 \, d\phi_0 - \frac{1}{c}\int_0^{2\pi} \int_0^{ct} \left[\int_0^\pi \hat{\boldsymbol{\theta}}_0 \sin\phi_0 \frac{\partial}{\partial \phi_0} J_r \, d\phi_0\right] dr_0 \, d\theta_0
\end{aligned}$$

Continue the simplification by integrating by parts.

$$\begin{aligned}
\mathbf{B}(x,y,z,t) &= \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0\times\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&\quad + \frac{1}{c}\int_0^{2\pi}\int_0^{ct}\left[\cancel{\hat{\mathbf{r}}_0(J_\theta\sin\phi_0)\Big|_0^{\pi}} - \int_0^{\pi}(J_\theta\sin\phi_0)\frac{\partial}{\partial\phi_0}\hat{\mathbf{r}}_0\,d\phi_0\right]dr_0\,d\theta_0 \\
&\quad - \frac{1}{c}\int_0^{\pi}\int_0^{ct}\left[\cancel{\hat{\mathbf{r}}_0J_\phi\Big|_0^{2\pi}} - \int_0^{2\pi}J_\phi\frac{\partial}{\partial\theta_0}\hat{\mathbf{r}}_0\,d\theta_0\right]dr_0\,d\phi_0 \\
&\quad + \frac{1}{c}\int_0^{\pi}\int_0^{ct}\left[\cancel{\hat{\boldsymbol{\phi}}_0J_r\Big|_0^{2\pi}} - \int_0^{2\pi}J_r\frac{\partial}{\partial\theta_0}\hat{\boldsymbol{\phi}}_0\,d\theta_0\right]dr_0\,d\phi_0 \\
&\quad - \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\left[\hat{\boldsymbol{\phi}}_0(r_0J_\theta)\Big|_0^{ct} - \int_0^{ct}(r_0J_\theta)\frac{\partial}{\partial r_0}\hat{\boldsymbol{\phi}}_0\,dr_0\right]\sin\phi_0\,d\theta_0\,d\phi_0 \\
&\quad + \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\left[\hat{\boldsymbol{\theta}}_0(r_0J_\phi)\Big|_0^{ct} - \int_0^{ct}(r_0J_\phi)\frac{\partial}{\partial r_0}\hat{\boldsymbol{\theta}}_0\,dr_0\right]\sin\phi_0\,d\theta_0\,d\phi_0 \\
&\quad - \frac{1}{c}\int_0^{2\pi}\int_0^{ct}\left[\cancel{(\hat{\boldsymbol{\theta}}_0\sin\phi_0)J_r\Big|_0^{\pi}} - \int_0^{\pi}J_r\frac{\partial}{\partial\phi_0}(\hat{\boldsymbol{\theta}}_0\sin\phi_0)\,d\phi_0\right]dr_0\,d\theta_0 \\
&= \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0\times\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&\quad + \frac{1}{c}\int_0^{2\pi}\int_0^{ct}\left[-\int_0^{\pi}(J_\theta\sin\phi_0)\hat{\boldsymbol{\phi}}_0\,d\phi_0\right]dr_0\,d\theta_0 \\
&\quad - \frac{1}{c}\int_0^{\pi}\int_0^{ct}\left[-\int_0^{2\pi}J_\phi(\hat{\boldsymbol{\theta}}_0\sin\phi_0)\,d\theta_0\right]dr_0\,d\phi_0 \\
&\quad \cancel{+ \frac{1}{c}\int_0^{\pi}\int_0^{ct}\left[-\int_0^{2\pi}J_r(\hat{\boldsymbol{\theta}}_0\cos\phi_0)\,d\theta_0\right]dr_0\,d\phi_0} \\
&\quad - \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\left[\hat{\boldsymbol{\phi}}_0\,ctJ_\theta(ct,\theta_0,\phi_0,0) - \cancel{\int_0^{ct}(r_0J_\theta)(0)\,dr_0}\right]\sin\phi_0\,d\theta_0\,d\phi_0 \\
&\quad + \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\left[\hat{\boldsymbol{\theta}}_0\,ctJ_\phi(ct,\theta_0,\phi_0,0) - \cancel{\int_0^{ct}(r_0J_\phi)(0)\,dr_0}\right]\sin\phi_0\,d\theta_0\,d\phi_0 \\
&\quad \cancel{- \frac{1}{c}\int_0^{2\pi}\int_0^{ct}\left\{-\int_0^{\pi}J_r(\hat{\boldsymbol{\theta}}_0\cos\phi_0)\,d\phi_0\right\}dr_0\,d\theta_0}
\end{aligned}$$

From equation (I) and the zero initial conditions, the components of $\mathbf{J}$ are all zero at $t=0$.

$$\frac{\partial \mathbf{E}}{\partial t}(x,y,z,0) = c\nabla\times\mathbf{B}(x,y,z,0) - 4\pi\mathbf{J}(x,y,z,0) \quad\rightarrow\quad \mathbf{0} = -4\pi\mathbf{J}(x,y,z,0) \quad\rightarrow\quad \mathbf{J}(x,y,z,0)=\mathbf{0}$$

That is, $J_\theta(ct,\theta_0,\phi_0,0)=0$ and $J_\phi(ct,\theta_0,\phi_0,0)=0$.

$$\begin{aligned}
\mathbf{B}(x,y,z,t) &= \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0\times\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&\quad - \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\hat{\boldsymbol{\phi}}_0\,J_\theta\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 + \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\hat{\boldsymbol{\theta}}_0\,J_\phi\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0
\end{aligned}$$

Write the last two integrals as one of a cross product of position and $\mathbf{J}$.

$$\begin{aligned}
\mathbf{B}(x,y,z,t) &= \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&\quad + \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[-\hat{\boldsymbol{\phi}}_0\,J_\theta\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) + \hat{\boldsymbol{\theta}}_0\,J_\phi\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right]\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&= \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&\quad + \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[(\hat{\mathbf{r}}_0\times\hat{\mathbf{r}}_0)J_r\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) + (\hat{\mathbf{r}}_0\times\hat{\boldsymbol{\theta}}_0)J_\theta\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right. \\
&\qquad\qquad \left. + (\hat{\mathbf{r}}_0\times\hat{\boldsymbol{\phi}}_0)J_\phi\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right]\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&= \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{c}\hat{\mathbf{r}}_0 \times \frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0 \sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&\quad + \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\hat{\mathbf{r}}_0\times\mathbf{J}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&= \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{r_0}\hat{\mathbf{r}}_0\times\mathbf{J}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) + \frac{1}{c}\hat{\mathbf{r}}_0\times\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&= \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{r_0^2}\mathbf{r}_0\times\mathbf{J}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) + \frac{1}{cr_0}\mathbf{r}_0\times\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&= \frac{1}{c}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct}\left[\frac{1}{r_0^3}\mathbf{r}_0\times\mathbf{J}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) + \frac{1}{cr_0^2}\mathbf{r}_0\times\frac{\partial \mathbf{J}}{\partial t}\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)\right] r_0^2\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0
\end{aligned}$$

Therefore, changing back to Cartesian coordinates,

$$\boxed{\begin{aligned}
\mathbf{B}(x,y,z,t) = \frac{1}{c}\iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2\le c^2t^2}} &\left\{\frac{(x_0-x)\hat{\mathbf{x}}+(y_0-y)\hat{\mathbf{y}}+(z_0-z)\hat{\mathbf{z}}}{\sqrt{[(x_0-x)^2+(y_0-y)^2+(z_0-z)^2]^3}}\right. \\
&\times\mathbf{J}\left(x_0,y_0,z_0,t-\frac{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}{c}\right) \\
&+\frac{1}{c}\frac{(x_0-x)\hat{\mathbf{x}}+(y_0-y)\hat{\mathbf{y}}+(z_0-z)\hat{\mathbf{z}}}{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2} \\
&\left.\times\frac{\partial \mathbf{J}}{\partial t}\left(x_0,y_0,z_0,t-\frac{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}{c}\right)\right\}dV_0.
\end{aligned}}$$

These boxed formulas for $\mathbf{E}$ and $\mathbf{B}$ are the fabled Jefimenko equations.