Exercise 1

(a) Use direct differentiation to check that (4) solves (3).

(b) Check directly that (5) solves the initial condition $u(x,0) = x^3$.

Solution

Part (a)

The PDE in (3) is

$$u_t + e^{x+t} u_x = 0. \tag{3}$$

We have to check that the solution in (4) is the correct one.

$$u(x,t) = f(e^{-x} + e^t) \tag{4}$$

Find the first derivatives of u with respect to t and x.

$$\begin{aligned} u_t &= e^t \cdot f' \\ u_x &= (-e^{-x}) \cdot f' \end{aligned}$$

Now plug these terms into the PDE.

$$u_t + e^{x+t} u_x = e^t f' + e^x e^t (-e^{-x}) f' = e^t f' - e^t f' = 0$$

Therefore, the general solution in (4) is the correct solution to the PDE in (3).

Part (b)

We have to check that the solution in (5) gives us the initial condition, $u(x,0) = x^3$, when we plug in $t = 0$.

$$u(x,t) = -[\ln(e^{-x} + e^t - 1)]^3 \tag{5}$$

Set $t = 0$.

$$u(x,0) = -[\ln(e^{-x} + \cancel{e^0} - \cancel{1})]^3$$

What remains is

$$u(x,0) = -[\ln(e^{-x})]^3.$$

Use the property of logarithms that allows us to turn the exponent of the argument to the coefficient.

$$u(x,0) = -(-x \ln e)^3 = -(-x)^3 = -(-x^3)$$

Therefore,

$$u(x,0) = x^3.$$