Exercise 4

Sketch some typical characteristic lines for Example 4.

Solution

In Example 4 the PDE under consideration is

$$u_t + uu_x = 0, \quad u(x,0) = x^2.$$

For a function of two variables $u = u(x,t)$, its differential is defined to be

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial t}\,dt.$$

Divide both sides by dt.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

This equation gives the relationship between the total derivative of u with respect to t and its partial derivatives. Comparing the right side with the given PDE, we see that on the curves (characteristics) in the xt-plane that satisfy

$$\frac{dx}{dt} = u, \tag{1}$$

the PDE simplifies to an ODE.

$$\frac{du}{dt} = 0, \tag{2}$$

u can be solved for by integrating both sides of equation (2) with respect to t.

$$u = f(\xi),$$

where f is an arbitrary function to be determined and ξ is a characteristic coordinate. Equation (2) tells us that u is independent of t, so the characteristic curves can be obtained by integrating both sides of equation (1) with respect to t.

$$x = ut + \xi$$

Solve this equation for ξ.

$$\xi = x - ut$$

The general solution to the PDE is then

$$u(x,t) = f(x - ut).$$

The aim now is to determine the particular function f that satisfies the initial condition.

$$u(x,0) = f(x) = x^2$$

Though this is in terms of x, the equation is actually $f(w) = w^2$, where w is any expression we choose: $f(x - ut) = (x - ut)^2$. Thus,

$$u(x,t) = (x - ut)^2.$$

Solve this equation for u.

$$u = (x - ut)^2$$

$$u = x^2 - 2xtu + t^2u^2$$

$$t^2u^2 - (1 + 2xt)u + x^2 = 0$$

Use the quadratic formula.

$$u(x,t) = \frac{1 + 2xt \pm \sqrt{1 + 4xt}}{2t^2}$$

The plus sign is omitted because in the limit that $t \to 0$, we require u to converge to x^2. Therefore,

$$u(x,t) = \frac{1 + 2xt - \sqrt{1 + 4xt}}{2t^2}.$$

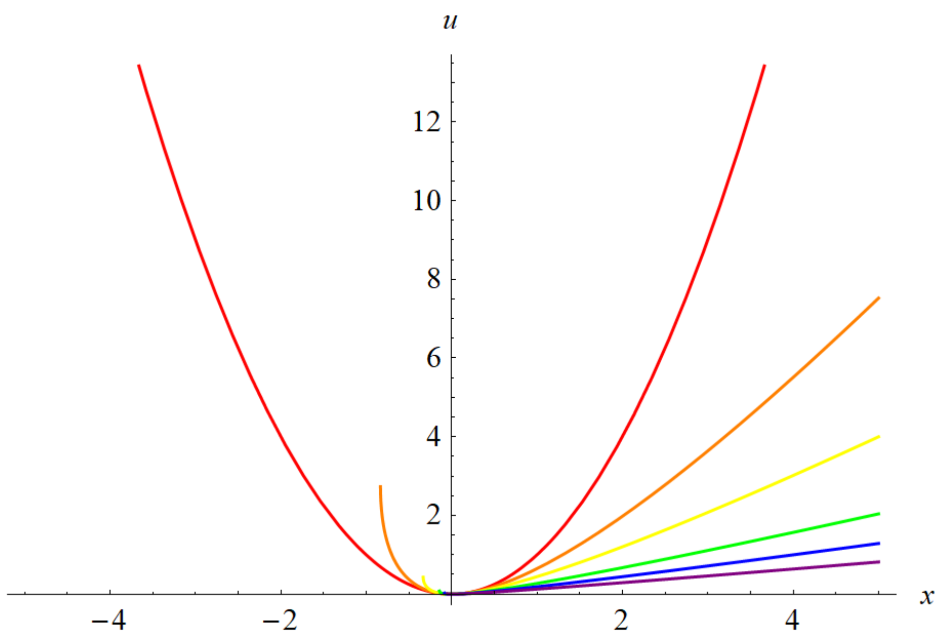

Figure 1: This is a plot of the solution u as a function of x for various times. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 0.3$, $t = 0.75$, $t = 1.75$, $t = 3$, $t = 5$, respectively.

The solution to the PDE starts as a quadratic polynomial $u = x^2$ and then decays to $u = 0$ for $x > 0$ as t goes to infinity.

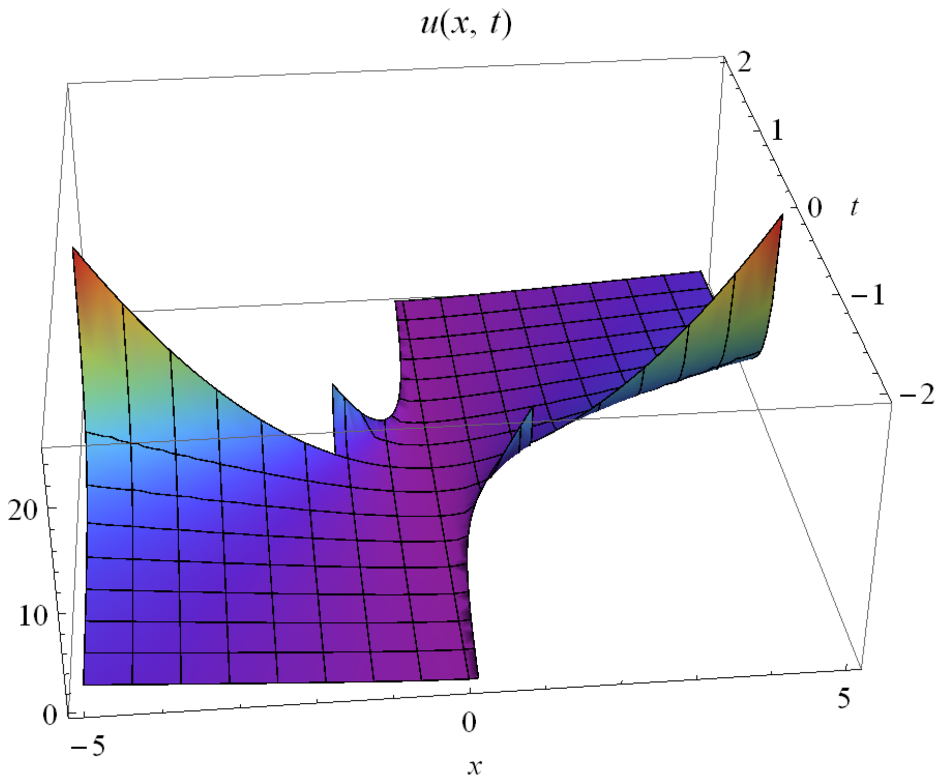

Figure 2: This is a plot of the two-dimensional solution surface $u(x, t)$ in three-dimensional space for $-5 < x < 5$ and $-2 < t < 2$.

Knowing the solution to the PDE is totally irrelevant for this problem, though. Since $u = f(\xi) = \xi^2$, the equation for the characteristics becomes

$$x = ut + \xi = \xi^2 t + \xi = \xi(\xi t + 1).$$

The way to plot them is to choose a certain value for ξ and then to plot the resulting equation in the xt-plane. This is done over and over until the plane is full. A computer can accomplish this task very efficiently. ξ is chosen here to go from -5 to 5, incrementing by 0.2 each time. The region in the xt-plane given by $1 + 4xt < 0$ has been blackened in the plots, as the solution to the PDE is not defined here. Notice that each of the characteristics are straight lines and that the closer its coordinate ξ is to 0, the more vertical its slope is.

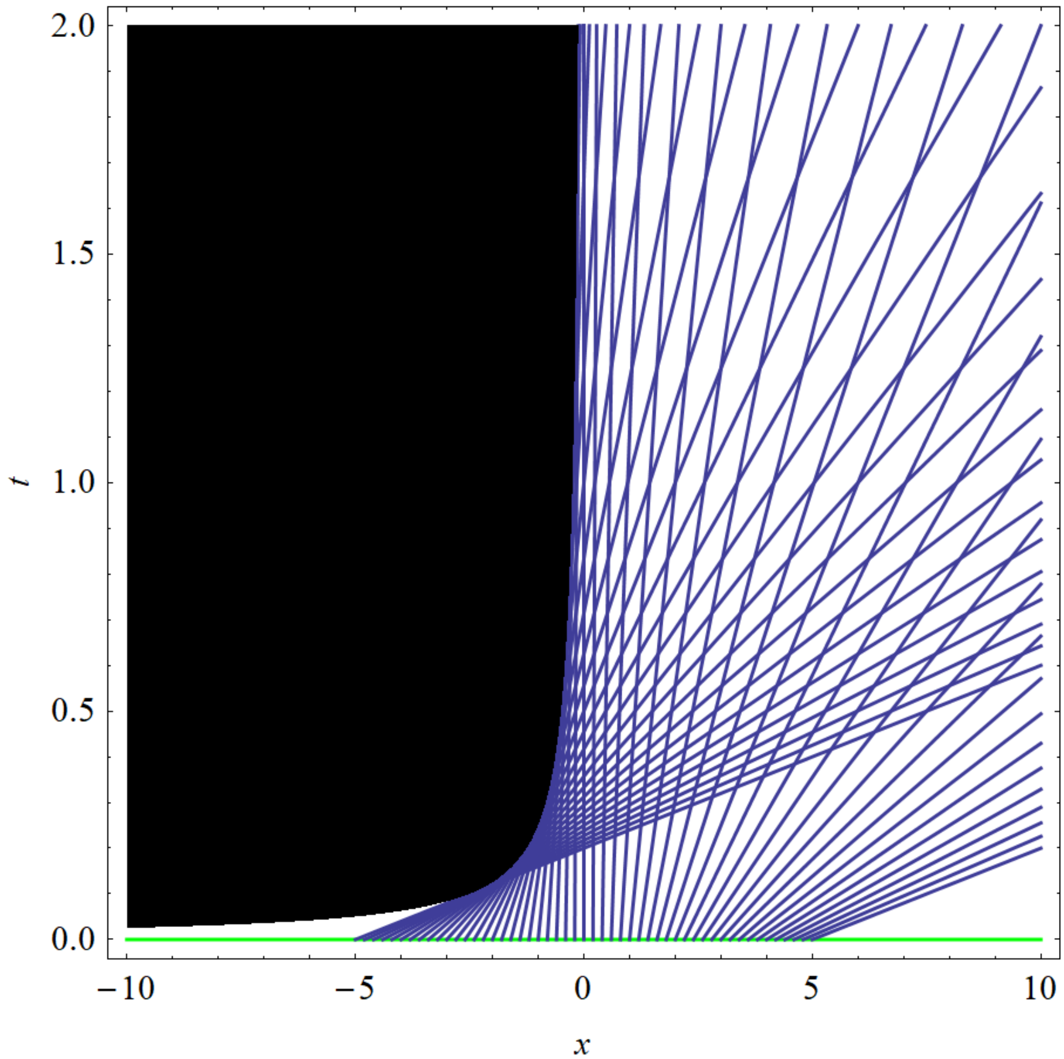

Figure 3: Plot of the characteristic curves in the top half of the xt-plane along with the given data curve ($t = 0$) in green. ξ is chosen here to go from -5 to 5, incrementing by 0.2 each time. The region in the xt-plane given by $1 + 4xt < 0$ has been blackened, as the solution to the PDE is not defined here.

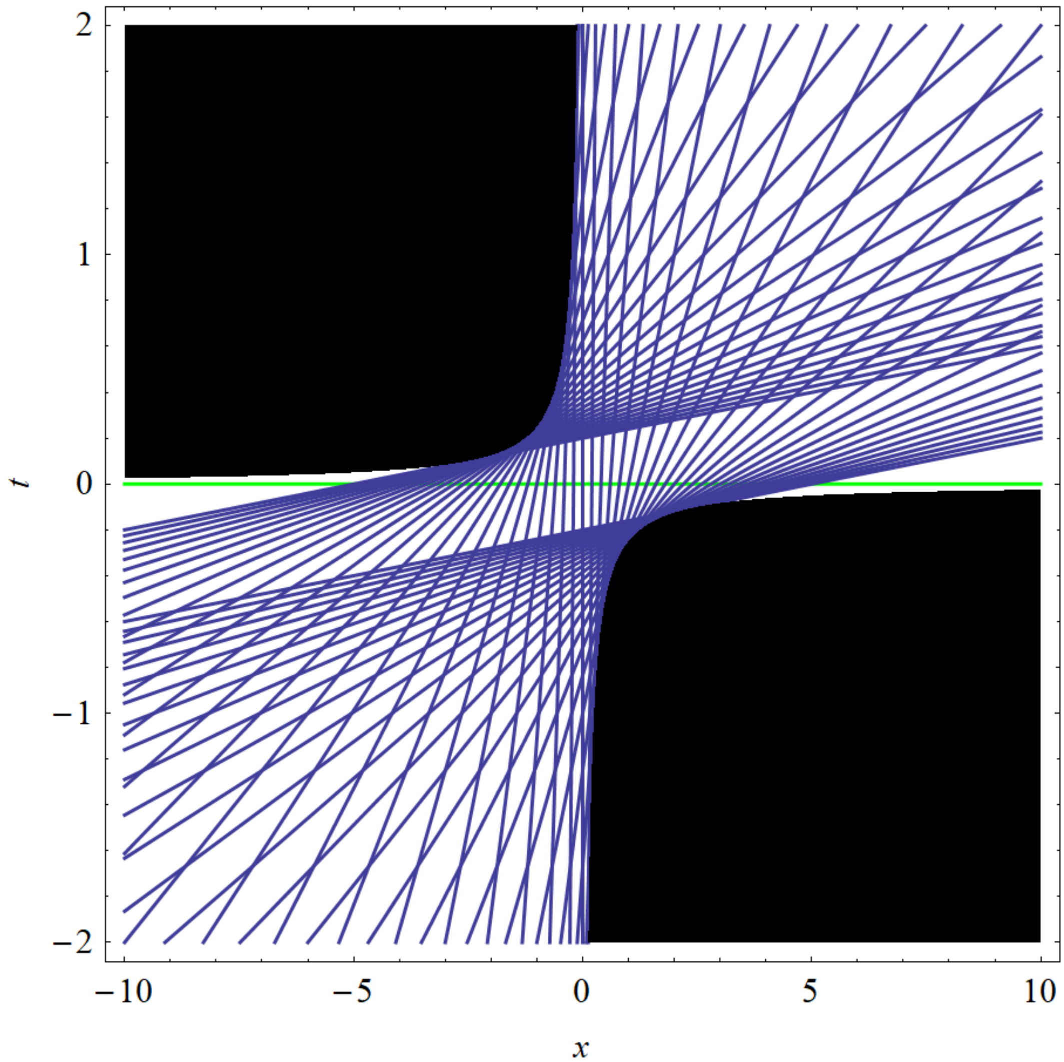

Figure 4: Plot of the characteristic curves in the xt-plane along with the given data curve ($t = 0$) in green. ξ is chosen here to go from -5 to 5, incrementing by 0.2 each time. The region in the xt-plane given by $1 + 4xt < 0$ has been blackened, as the solution to the PDE is not defined here.