Exercise 5

Solve $u_t + u^2 u_x = 0$ with $u(x, 0) = 2 + x$.

Solution

For a function of two variables $u = u(x, t)$, its differential is defined to be

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial t}\,dt.$$

Divide both sides by dt.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

This equation gives the relationship between the total derivative of u with respect to t and its partial derivatives. Comparing the right side with the given PDE, we see that on the curves (characteristics) in the xt-plane that satisfy

$$\frac{dx}{dt} = u^2, \tag{1}$$

the PDE simplifies to an ODE.

$$\frac{du}{dt} = 0, \tag{2}$$

u can be solved for by integrating both sides of equation (2) with respect to t.

$$u = f(\xi),$$

where f is an arbitrary function to be determined and ξ is a characteristic coordinate. Equation (2) tells us that u is independent of t, so the characteristic curves can be obtained by integrating both sides of equation (1) with respect to t.

$$x = u^2 t + \xi$$

Solve this equation for ξ.

$$\xi = x - u^2 t$$

The general solution to the PDE is then

$$u(x, t) = f(x - u^2 t).$$

The aim now is to determine the particular function f that satisfies the initial condition.

$$u(x, 0) = f(x) = 2 + x$$

Though this is in terms of x, the equation is actually $f(w) = 2 + w$, where w is any expression we choose: $f(x - u^2 t) = 2 + x - u^2 t$. Thus,

$$u(x, t) = 2 + x - u^2 t.$$

Solve this equation for u.

$$u = 2 + x - u^2 t$$

$$tu^2 + u - (2 + x) = 0$$

Use the quadratic formula.

$$u(x,t) = \frac{-1 \pm \sqrt{1 + 4t(2 + x)}}{2t}$$

The minus sign is omitted because in the limit that $t \to 0$, we require u to converge to $2 + x$. Therefore,

$$\boxed{u(x,t) = \frac{-1 + \sqrt{1 + 4t(2 + x)}}{2t},}$$

where the solution is defined for $1 + 4t(2 + x) \geq 0$ and of course $t \neq 0$.

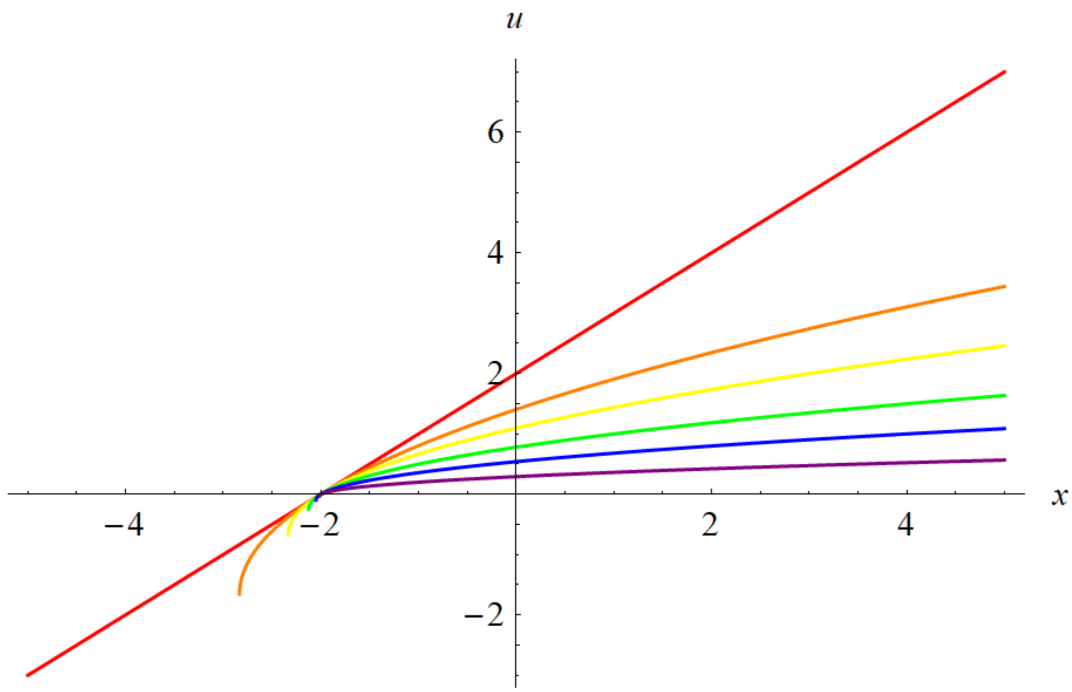

Figure 1: This is a plot of the solution u as a function of x for various times. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 0.3$, $t = 0.75$, $t = 2$, $t = 5$, $t = 20$, respectively.

The solution to the PDE starts as a linear function $u = x + 2$ and then decays to $u = 0$ for $x > -2$ as t goes to infinity.

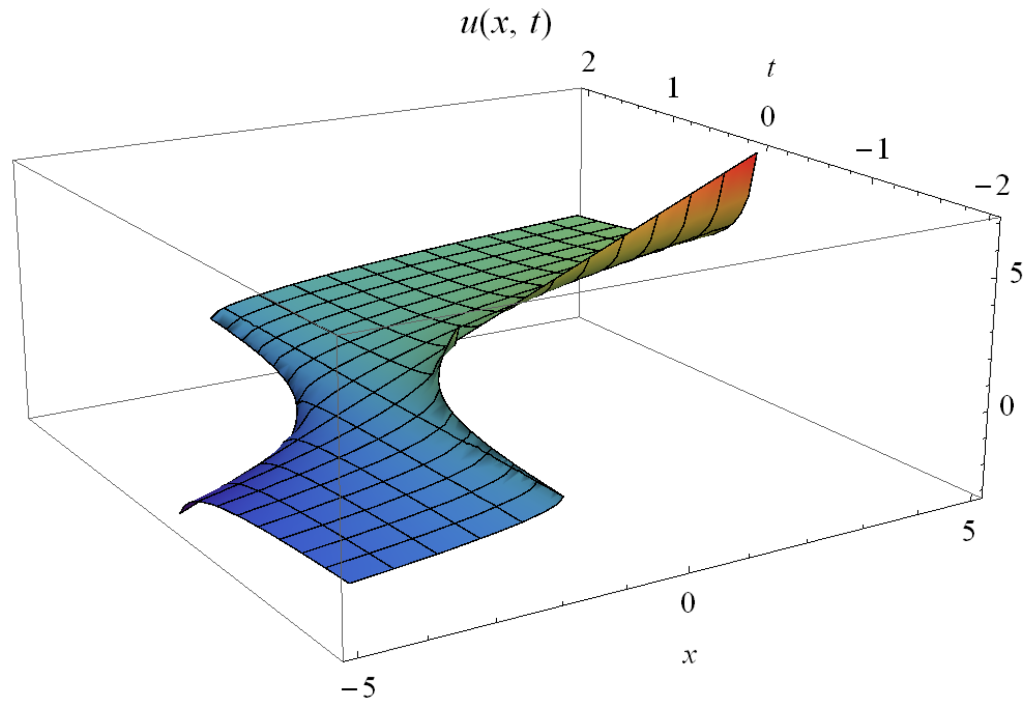

Figure 2: This is a plot of the two-dimensional solution surface $u(x,t)$ in three-dimensional space for $-5 < x < 5$ and $-2 < t < 2$.

Now the characteristic curves will be plotted. Since $u = f(\xi) = 2 + \xi$, the equation for the characteristics becomes

$$x = u^2 t + \xi = (2+\xi)^2 t + \xi.$$

The way to plot them is to choose a certain value for ξ and then to plot the resulting equation in the xt-plane. This is done over and over until the plane is full. A computer can accomplish this task very efficiently. Notice that all the characteristics are straight lines and that they become increasingly vertical as ξ tends to -2.

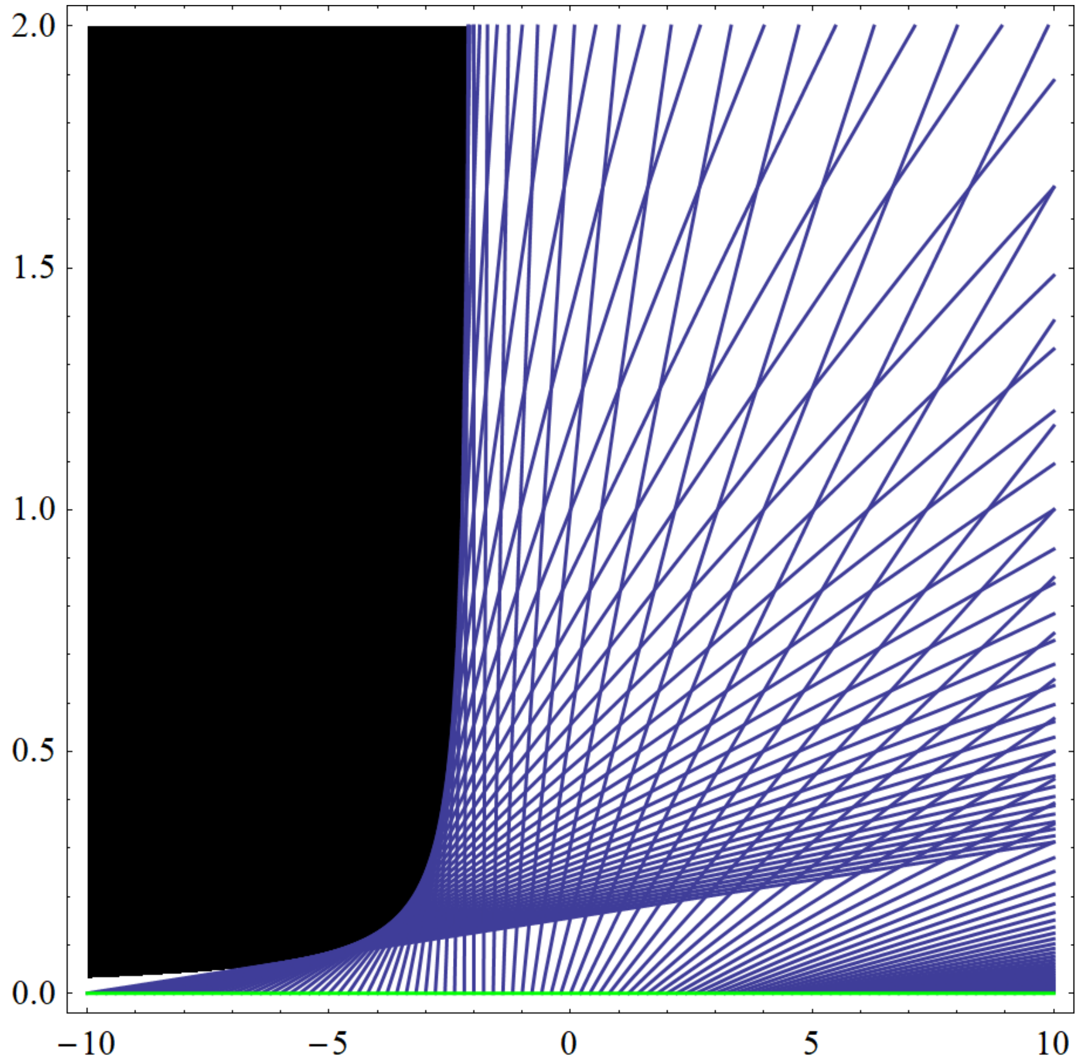

Figure 3: Plot of the characteristic curves in the upper half of the xt-plane along with the given data curve ($t = 0$) in green. ξ is chosen here to go from -10 to 10, incrementing by 0.25 each time. The region in the xt-plane given by $1 + 4t(2 + x) < 0$ has been blackened, as the solution to the PDE is not defined here.

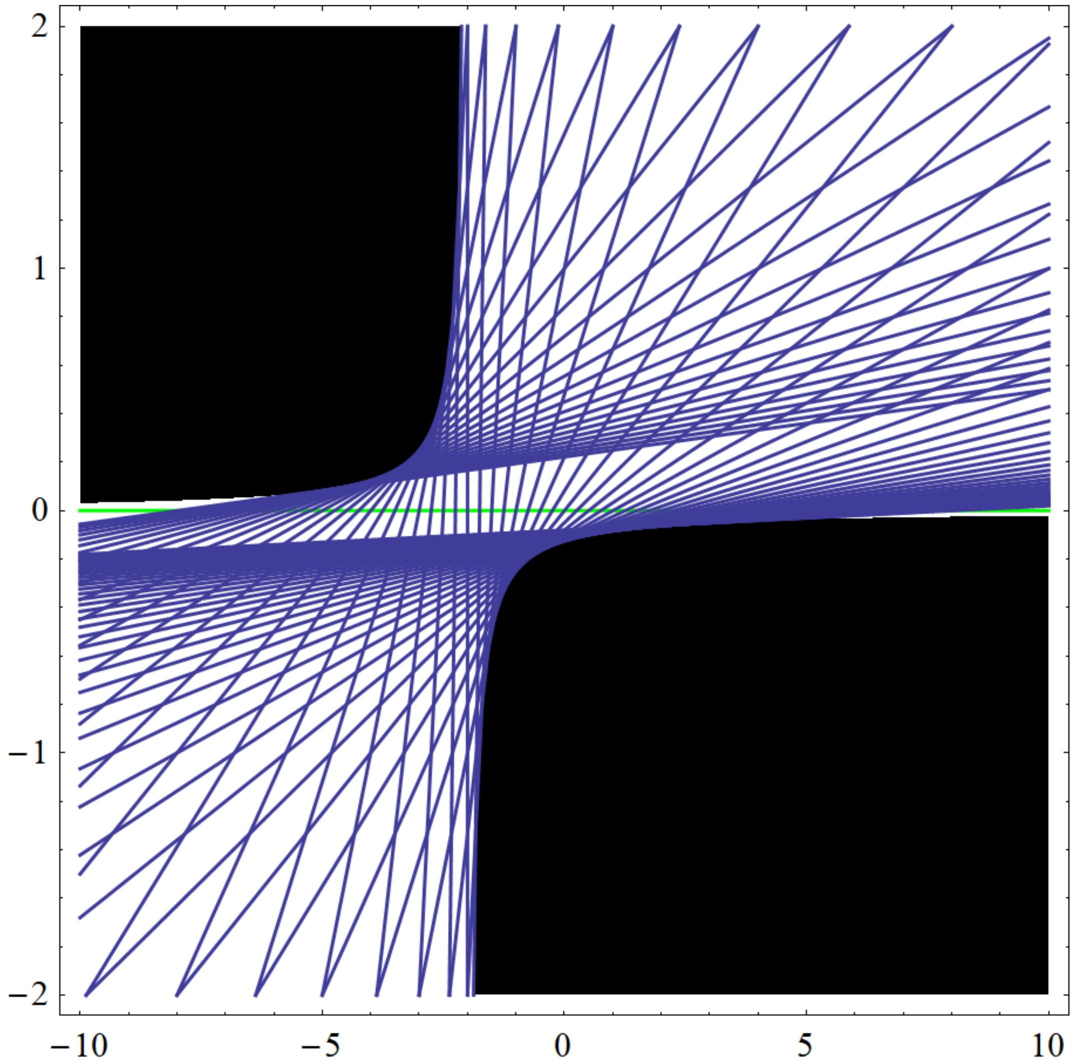

Figure 4: Plot of the characteristic curves in the xt-plane along with the given data curve ($t = 0$) in green. ξ is chosen here to go from -8 to 8, incrementing by 0.25 each time. The region in the xt-plane given by $1 + 4t(2 + x) < 0$ has been blackened, as the solution to the PDE is not defined here.