Exercise 9

Check by direct differentiation that the formula $u(x,t) = \phi(z)$, where z is given implicitly by $x - z = ta(\phi(z))$, does indeed provide a solution of the PDE (1).

Solution

The PDE in (1) is as follows.

$$u_t + a(u)u_x = 0 \tag{1}$$

The solution we intend to verify is $u(x,t) = \phi(z)$, where $z = z(x,t)$. Use the chain rule to determine expressions for u_t and u_x.

$$\begin{aligned} u_t &= \frac{\partial}{\partial t}\phi(z) = \frac{d\phi}{dz}\frac{\partial z}{\partial t} \\ u_x &= \frac{\partial}{\partial x}\phi(z) = \frac{d\phi}{dz}\frac{\partial z}{\partial x} \end{aligned}$$

Our aim now is to determine $\partial z/\partial t$ and $\partial z/\partial x$ by differentiating the given implicit relationship.

$$x - z = ta[\phi(z)]$$

Differentiate both sides of it with respect to t.

$$-\frac{\partial z}{\partial t} = a + ta'\phi'\frac{\partial z}{\partial t}$$

Differentiate both sides of it with respect to x.

$$1 - \frac{\partial z}{\partial x} = ta'\phi'\frac{\partial z}{\partial x}$$

Solve the above equations for $\partial z/\partial t$ and $\partial z/\partial x$.

$$\begin{aligned} \frac{\partial z}{\partial t} &= -\frac{a}{1 + ta'\phi'} \\ \frac{\partial z}{\partial x} &= \frac{1}{1 + ta'\phi'} \end{aligned}$$

So we have

$$\begin{aligned} u_t + a(u)u_x &= u_t + au_x \\ &= \frac{d\phi}{dz}\frac{\partial z}{\partial t} + a\frac{d\phi}{dz}\frac{\partial z}{\partial x} \\ &= \frac{d\phi}{dz}\left(-\frac{a}{1 + ta'\phi'}\right) + a\frac{d\phi}{dz}\left(\frac{1}{1 + ta'\phi'}\right) \\ &= -\frac{d\phi}{dz}\left(\frac{a}{1 + ta'\phi'}\right) + \frac{d\phi}{dz}\left(\frac{a}{1 + ta'\phi'}\right) \\ &= 0. \end{aligned}$$

Therefore, $u(x,t) = \phi(z)$ is a solution of the PDE (1), where z is given implicitly by $x - z = ta[\phi(z)]$.