Exercise 12

Solve $u_t + uu_x = 1$ with $u(x,0) = x$.

Solution

For a function of two variables $u = u(x,t)$, its differential is defined to be

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial t}\,dt.$$

Divide both sides by dt.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

This equation gives the relationship between the total derivative of u with respect to t and its partial derivatives. Comparing the right side with the given PDE, we see that on the curves (characteristics) in the xt-plane that satisfy

$$\frac{dx}{dt} = u, \tag{1}$$

the PDE simplifies to an ODE.

$$\frac{du}{dt} = 1, \tag{2}$$

u can be solved for by integrating both sides of equation (2) with respect to t.

$$u = t + f(\xi),$$

where f is an arbitrary function to be determined and ξ is a characteristic coordinate. Substitute this result into equation (1).

$$\frac{dx}{dt} = t + f(\xi)$$

Integrate both sides of this equation with respect to t.

$$x = \frac{t^2}{2} + f(\xi)t + \xi$$

Now take steps to replace $f(\xi)$ with u.

$$\begin{aligned} x &= t\left[\frac{t}{2} + f(\xi)\right] + \xi \\ &= t\left[t + f(\xi) - \frac{t}{2}\right] + \xi \end{aligned}$$

Substitute u back into the equation.

$$x = t\left(u - \frac{t}{2}\right) + \xi \tag{3}$$

Now solve for ξ.

$$\xi = x - t\left(u - \frac{t}{2}\right)$$

Thus, the general solution to the PDE is

$$u(x,t) = t + f\left[x - t\left(u - \frac{t}{2}\right)\right].$$

The aim now is to determine the particular function f that satisfies the initial condition.

$$u(x,0) = f(x) = x$$

Though this is in terms of x, the equation is actually $f(w) = w$, where w is any expression we choose:

$$f\left[x - t\left(u - \frac{t}{2}\right)\right] = x - t\left(u - \frac{t}{2}\right).$$

So we have for the solution

$$u(x,t) = t + x - t\left(u - \frac{t}{2}\right).$$

Solve this for u.

$$\begin{aligned} u &= t + x - tu + \frac{t^2}{2} \\ u + tu &= t + x + \frac{t^2}{2} \\ u(1+t) &= x + t + \frac{t^2}{2} \\ u &= \frac{x + t + \frac{t^2}{2}}{1+t} \end{aligned}$$

Therefore,

$$\boxed{u(x,t) = \frac{2(x+t) + t^2}{2(1+t)}.}$$

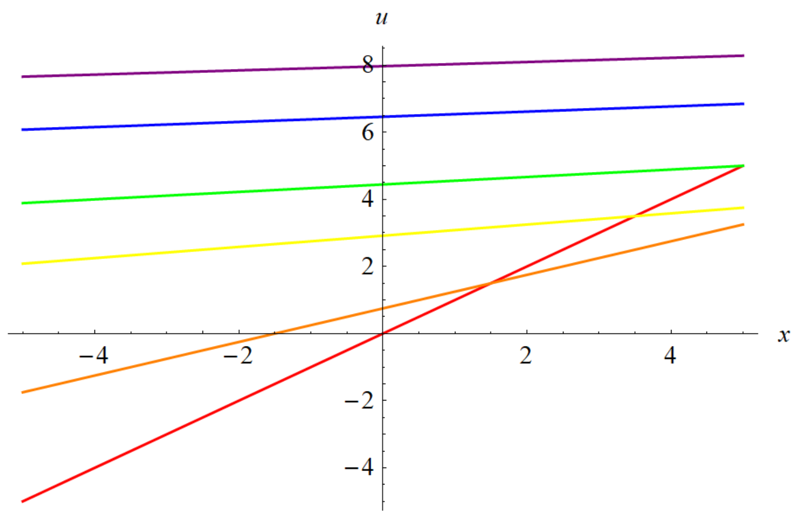

Figure 1: This is a plot of the solution u as a function of x for various times. The curves in red, orange, yellow, green, blue, and purple correspond to $t = 0$, $t = 1$, $t = 5$, $t = 8$, $t = 12$, $t = 15$, respectively.

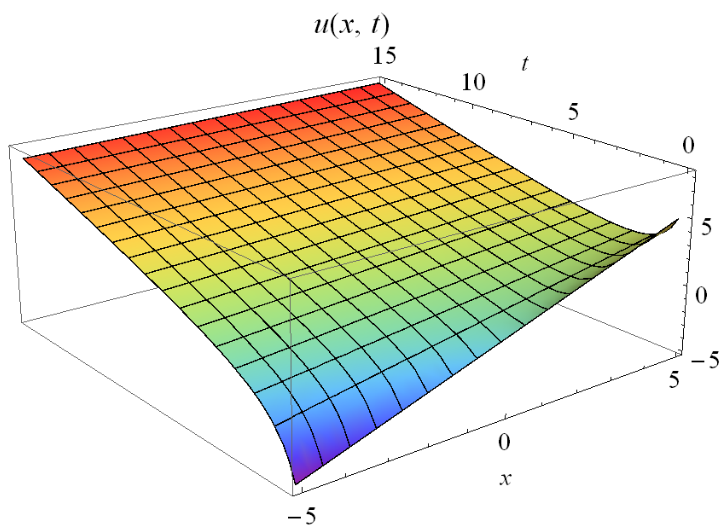

Figure 2: This is a plot of the two-dimensional solution surface $u(x, t)$ in three-dimensional space for $-5 < x < 5$ and $0 < t < 15$.

To determine where in the xt-plane the solution to the PDE is valid, it's necessary to plot the characteristic curves along with the given data curve at $t = 0$. It was determined earlier that $u = t + f(\xi) = t + \xi$, so equation (3) for the characteristics becomes

$$x = t\left(u - \frac{t}{2}\right) + \xi$$
$$x = t\left(\xi + \frac{t}{2}\right) + \xi.$$

The way to plot them is to choose a certain value for ξ and then to plot the resulting equation in the xt-plane. This is done over and over until the plane is full. A computer can accomplish this task very efficiently.

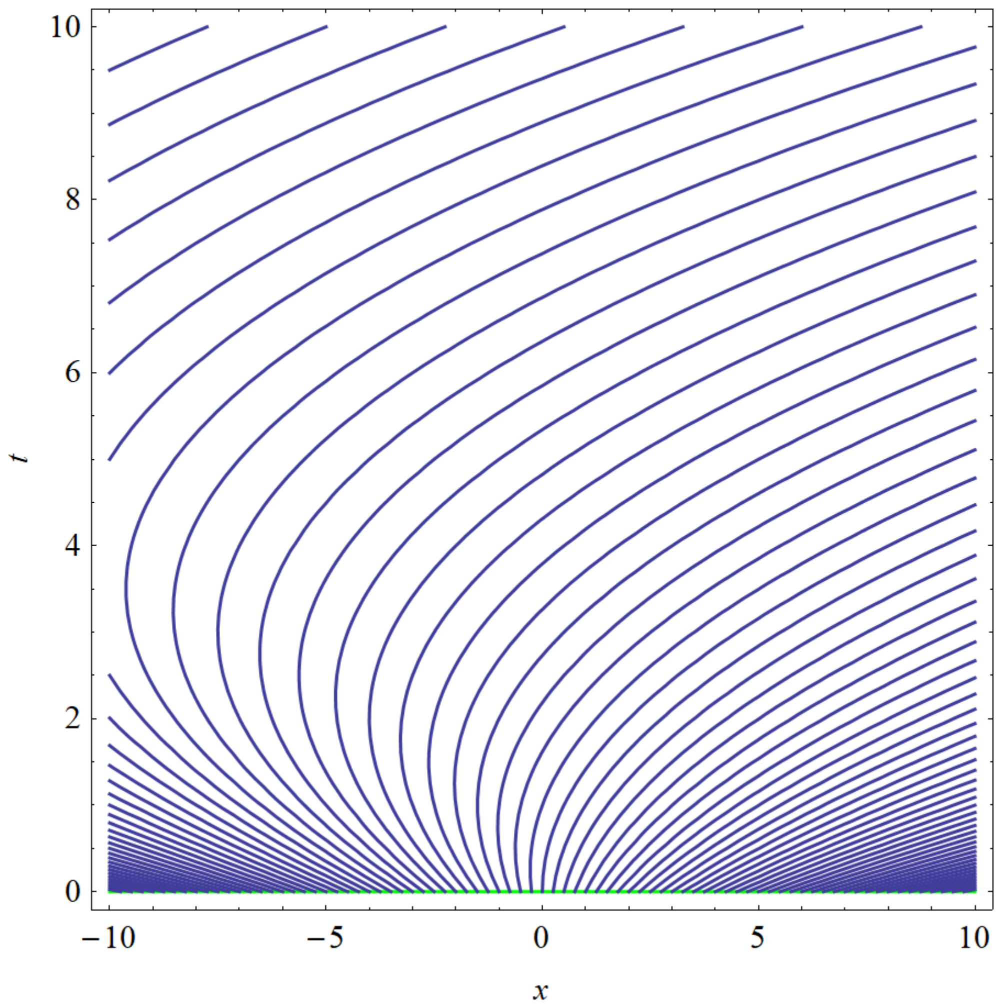

Figure 3: Plot of the characteristic curves in the upper half of the xt-plane along with the given data curve ($t = 0$) in green. ξ is chosen here to go from -10 to 10, incrementing by 0.25 each time.

Since none of the characteristics cross in the upper half of the xt-plane, no shock waves develop here. Because the characteristic curves cover this entire region and the data curve intersects all of them exactly once, the boxed solution is valid for all x and $t > 0$.