

Exercise 1

Solve $u_{tt} = c^2 u_{xx}$, $u(x, 0) = e^x$, $u_t(x, 0) = \sin x$.

Solution

Solution by Operator Factorization

To solve the wave equation on the whole line ($-\infty < x < \infty$), the method of operator factorization can be used.

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= 0 \\ \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u &= 0 \\ \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u &= 0\end{aligned}$$

If we let

$$v = \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) u,$$

then the previous equation becomes

$$\left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) v = 0.$$

In other words, the method of operator factorization reduces the wave equation to a system of first-order PDEs.

$$\begin{cases} \frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = v \\ \frac{\partial v}{\partial t} + c \frac{\partial v}{\partial x} = 0 \end{cases}$$

Note that the differential of a two-dimensional function $h(x, t)$ is defined as

$$dh = \frac{\partial h}{\partial t} dt + \frac{\partial h}{\partial x} dx$$

Dividing both sides by dt results in the fundamental relationship between the total derivative of h and its partial derivatives.

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{dx}{dt} \frac{\partial h}{\partial x}$$

Along the curves in the tx -plane defined by

$$\frac{dx}{dt} = c, \quad x(\xi, 0) = \xi, \tag{1}$$

where ξ is a characteristic coordinate, the PDE for v becomes an ODE.

$$\frac{dv}{dt} = 0 \tag{2}$$

Integrate both sides of equation (2) with respect to t .

$$v(\xi, t) = f(\xi)$$

Here f is an arbitrary function of the characteristic coordinate. Now integrate both sides of equation (1) with respect to t .

$$x = ct + \xi \quad \rightarrow \quad \xi = x - ct$$

In terms of the original variables, then,

$$v(x, t) = f(x - ct).$$

Consequently, the PDE for u is

$$\frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = f(x - ct).$$

Along the curves in the tx -plane defined by

$$\frac{dx}{dt} = -c, \quad x(\eta, 0) = \eta, \tag{3}$$

where η is another characteristic coordinate, the PDE for u becomes an ODE.

$$\frac{du}{dt} = f(x - ct) \tag{4}$$

Integrate both sides of equation (3) with respect to t .

$$x = -ct + \eta \quad \rightarrow \quad \eta = x + ct$$

Eliminate x from equation (4).

$$\frac{du}{dt} = f(-2ct + \eta)$$

Integrate both sides with respect to t .

$$u(\eta, t) = F(-2ct + \eta) + G(\eta)$$

Here F and G are arbitrary functions. Now that u is known, change back to the original variables to obtain the general solution.

$$u(x, t) = F(x - ct) + G(x + ct)$$

Apply the initial conditions to determine F and G .

$$\begin{aligned} u(x, 0) = e^x &\quad \rightarrow \quad F(x) + G(x) = e^x \\ u_t(x, 0) = \sin x &\quad \rightarrow \quad -cF'(x) + cG'(x) = \sin x \end{aligned}$$

Multiply both sides of the first equation by c and differentiate both sides with respect to x .

$$\begin{aligned} cF'(x) + cG'(x) &= ce^x \\ -cF'(x) + cG'(x) &= \sin x \end{aligned}$$

Adding the respective sides yields

$$2cG'(x) = ce^x + \sin x,$$

whereas subtracting the respective sides yields

$$2cF'(x) = ce^x - \sin x.$$

As a result,

$$F'(x) = \frac{1}{2}e^x - \frac{1}{2c} \sin x$$

$$G'(x) = \frac{1}{2}e^x + \frac{1}{2c} \sin x.$$

Integrate these functions to obtain $F(x)$ and $G(x)$.

$$F(x) = \frac{1}{2}e^x + \frac{1}{2c} \cos x$$

$$G(x) = \frac{1}{2}e^x - \frac{1}{2c} \cos x$$

What we solved for are actually $F(w)$ and $G(w)$, where w is any expression we wish.

$$F(x - ct) = \frac{1}{2}e^{x-ct} + \frac{1}{2c} \cos(x - ct)$$

$$G(x + ct) = \frac{1}{2}e^{x+ct} - \frac{1}{2c} \cos(x + ct)$$

Therefore,

$$\begin{aligned} u(x, t) &= F(x - ct) + G(x + ct) \\ &= \frac{1}{2}e^{x-ct} + \frac{1}{2c} \cos(x - ct) + \frac{1}{2}e^{x+ct} - \frac{1}{2c} \cos(x + ct) \\ &= \frac{1}{2}(e^{x-ct} + e^{x+ct}) + \frac{1}{2c}[\cos(x - ct) - \cos(x + ct)] \\ &= \frac{e^x}{2}(e^{-ct} + e^{ct}) + \frac{1}{2c}[(\cos x \cos ct + \sin x \sin ct) - (\cos x \cos ct - \sin x \sin ct)] \\ &= e^x \cosh ct + \frac{1}{2c}(2 \sin x \sin ct) \\ &= e^x \cosh ct + \frac{1}{c} \sin x \sin ct. \end{aligned}$$

Solution by the Method of Characteristics

$$u_{tt} - c^2 u_{xx} = 0$$

Comparing this equation with the general form of a second-order PDE, $Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G$, we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2A} \left(B \pm \sqrt{B^2 - 4AC} \right) \\ \frac{dx}{dt} &= \frac{1}{2} \left(\pm \sqrt{0 + 4c^2} \right) \\ \frac{dx}{dt} &= c \quad \text{or} \quad \frac{dx}{dt} = -c.\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 4c^2$, is greater than 0, which means that the PDE is hyperbolic. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the tx -plane.

$$x = ct + C_1 \quad \text{or} \quad x = -ct + C_2.$$

Solve for the constants of integration.

$$C_1 = x - ct$$

$$C_2 = x + ct$$

Make the change of variables, $\xi = x - ct$ and $\eta = x + ct$, so that the PDE takes the simplest form. Use the chain rule to determine the old derivatives in terms of these new variables.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t} = u_\xi(-c) + u_\eta(c) = c(u_\eta - u_\xi)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) = \left(\frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta} \right) [c(u_\eta - u_\xi)] = \left(-c \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial \eta} \right) [c(u_\eta - u_\xi)] = c^2(u_{\eta\eta} - 2u_{\xi\eta} + u_{\xi\xi})$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = u_\xi(1) + u_\eta(1) = u_\eta + u_\xi$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \left(\frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} \right) (u_\eta + u_\xi) = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) (u_\eta + u_\xi) = u_{\eta\eta} + 2u_{\xi\eta} + u_{\xi\xi}$$

Substitute these formulas into the PDE and simplify.

$$\begin{aligned}c^2(u_{\eta\eta} - 2u_{\xi\eta} + u_{\xi\xi}) - c^2(u_{\eta\eta} + 2u_{\xi\eta} + u_{\xi\xi}) &= 0 \\ -4c^2 u_{\xi\eta} &= 0 \\ u_{\xi\eta} &= 0\end{aligned}$$

This is known as the first canonical form of the wave equation. Integrate both sides with respect to η .

$$u_\xi = f(\xi)$$

Here f is an arbitrary function of ξ . Integrate both sides with respect to ξ to get u .

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

F and G are arbitrary functions of ξ and η , respectively. Now that u is known, change back to the original variables.

$$u(x, t) = F(x - ct) + G(x + ct)$$

Apply the initial conditions to determine F and G .

$$\begin{aligned} u(x, 0) = e^x &\rightarrow F(x) + G(x) = e^x \\ u_t(x, 0) = \sin x &\rightarrow -cF'(x) + cG'(x) = \sin x \end{aligned}$$

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As a result,

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Integrate these functions to obtain $F(x)$ and $G(x)$.

$$\begin{aligned} F(x) &= \frac{1}{2}e^x + \frac{1}{2c}\cos x \\ G(x) &= \frac{1}{2}e^x - \frac{1}{2c}\cos x \end{aligned}$$

What we solved for are actually $F(w)$ and $G(w)$, where w is any expression we wish.

$$\begin{aligned} F(x - ct) &= \frac{1}{2}e^{x-ct} + \frac{1}{2c}\cos(x - ct) \\ G(x + ct) &= \frac{1}{2}e^{x+ct} - \frac{1}{2c}\cos(x + ct) \end{aligned}$$

Therefore,

$$\begin{aligned} u(x, t) &= F(x - ct) + G(x + ct) \\ &= \frac{1}{2}e^{x-ct} + \frac{1}{2c}\cos(x - ct) + \frac{1}{2}e^{x+ct} - \frac{1}{2c}\cos(x + ct) \\ &= \frac{e^x}{2}(e^{-ct} + e^{ct}) + \frac{1}{2c}[(\cos x \cos ct + \sin x \sin ct) - (\cos x \cos ct - \sin x \sin ct)] \\ &= e^x \cosh ct + \frac{1}{c}\sin x \sin ct. \end{aligned}$$