Exercise 2

Solve $u_{tt} = c^2 u_{xx},\ u(x,0) = \log(1+x^2),\ u_t(x,0) = 4 + x.$

Solution

Solution by Operator Factorization

To solve the wave equation on the whole line ($-\infty < x < \infty$), the method of operator factorization can be used.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = 0$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0$$

If we let

$$v = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u,$$

then the previous equation becomes

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) v = 0.$$

In other words, the method of operator factorization reduces the wave equation to a system of first-order PDEs.

$$\begin{cases} \dfrac{\partial u}{\partial t} - c\dfrac{\partial u}{\partial x} = v \\ \dfrac{\partial v}{\partial t} + c\dfrac{\partial v}{\partial x} = 0 \end{cases}$$

Note that the differential of a two-dimensional function $h(x,t)$ is defined as

$$dh = \frac{\partial h}{\partial t}\,dt + \frac{\partial h}{\partial x}\,dx$$

Dividing both sides by dt results in the fundamental relationship between the total derivative of h and its partial derivatives.

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{dx}{dt}\frac{\partial h}{\partial x}$$

Along the curves in the tx-plane defined by

$$\frac{dx}{dt} = c, \quad x(\xi, 0) = \xi, \tag{1}$$

where ξ is a characteristic coordinate, the PDE for v becomes an ODE.

$$\frac{dv}{dt} = 0 \tag{2}$$

Integrate both sides of equation (2) with respect to t.

$$v(\xi, t) = f(\xi)$$

Here f is an arbitrary function of the characteristic coordinate. Now integrate both sides of equation (1) with respect to t.

$$x = ct + \xi \quad \rightarrow \quad \xi = x - ct$$

In terms of the original variables, then,

$$v(x,t) = f(x - ct).$$

Consequently, the PDE for u is

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = f(x - ct).$$

Along the curves in the tx-plane defined by

$$\frac{dx}{dt} = -c, \quad x(\eta, 0) = \eta, \tag{3}$$

where η is another characteristic coordinate, the PDE for u becomes an ODE.

$$\frac{du}{dt} = f(x - ct) \tag{4}$$

Integrate both sides of equation (3) with respect to t.

$$x = -ct + \eta \quad \rightarrow \quad \eta = x + ct$$

Eliminate x from equation (4).

$$\frac{du}{dt} = f(-2ct + \eta)$$

Integrate both sides with respect to t.

$$u(\eta, t) = F(-2ct + \eta) + G(\eta)$$

Here F and G are arbitrary functions. Now that u is known, change back to the original variables to obtain the general solution.

$$u(x,t) = F(x - ct) + G(x + ct)$$

Apply the initial conditions to determine F and G.

$$\begin{aligned} u(x,0) = \log(1 + x^2) \quad &\rightarrow \quad F(x) + G(x) = \log(1 + x^2) \\ u_t(x,0) = 4 + x \quad &\rightarrow \quad -cF'(x) + cG'(x) = 4 + x \end{aligned}$$

Multiply both sides of the first equation by c and differentiate both sides with respect to x.

$$\begin{aligned} cF'(x) + cG'(x) &= c\frac{2x}{1 + x^2} \\ -cF'(x) + cG'(x) &= 4 + x \end{aligned}$$

Adding the respective sides yields

$$2cG'(x) = c\frac{2x}{1 + x^2} + 4 + x,$$

whereas subtracting the respective sides yields

$$2cF'(x) = c\frac{2x}{1+x^2} - 4 - x.$$

As a result,

$$\begin{aligned} F'(x) &= \frac{1}{2}\frac{2x}{1+x^2} - \frac{1}{2c}(4+x) \\ G'(x) &= \frac{1}{2}\frac{2x}{1+x^2} + \frac{1}{2c}(4+x). \end{aligned}$$

Integrate these functions to obtain $F(x)$ and $G(x)$.

$$\begin{aligned} F(x) &= \frac{1}{2}\log(1+x^2) - \frac{1}{4c}(8x+x^2) \\ G(x) &= \frac{1}{2}\log(1+x^2) + \frac{1}{4c}(8x+x^2) \end{aligned}$$

What we solved for are actually $F(w)$ and $G(w)$, where w is any expression we wish.

$$\begin{aligned} F(x-ct) &= \frac{1}{2}\log[1+(x-ct)^2] - \frac{1}{4c}[8(x-ct)+(x-ct)^2] \\ G(x+ct) &= \frac{1}{2}\log[1+(x+ct)^2] + \frac{1}{4c}[8(x+ct)+(x+ct)^2] \end{aligned}$$

Therefore,

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \\ &= \frac{1}{2}\log[1+(x-ct)^2] - \frac{1}{4c}[8(x-ct)+(x-ct)^2] + \frac{1}{2}\log[1+(x+ct)^2] + \frac{1}{4c}[8(x+ct)+(x+ct)^2] \\ &= \frac{1}{2}\left\{\log[1+(x-ct)^2] + \log[1+(x+ct)^2]\right\} + \frac{1}{4c}[8(x+ct)+(x+ct)^2 - 8(x-ct) - (x-ct)^2] \\ &= \frac{1}{2}\log[1+(x-ct)^2][1+(x+ct)^2] + \frac{1}{4c}(16ct + 4ctx) \\ &= \log\sqrt{[1+(x-ct)^2][1+(x+ct)^2]} + t(4+x). \end{aligned}$$

Solution by the Method of Characteristics

$$u_{tt} - c^2 u_{xx} = 0$$

Comparing this equation with the general form of a second-order PDE,
$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G$, we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dx}{dt} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dx}{dt} = \frac{1}{2}\left(\pm\sqrt{0 + 4c^2}\right)$$
$$\frac{dx}{dt} = c \quad \text{or} \quad \frac{dx}{dt} = -c.$$

Note that the discriminant, $B^2 - 4AC = 4c^2$, is greater than 0, which means that the PDE is hyperbolic. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the tx-plane.

$$x = ct + C_1 \quad \text{or} \quad x = -ct + C_2.$$

Solve for the constants of integration.

$$C_1 = x - ct$$
$$C_2 = x + ct$$

Make the change of variables, $\xi = x - ct$ and $\eta = x + ct$, so that the PDE takes the simplest form. Use the chain rule to determine the old derivatives in terms of these new variables.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = u_\xi(-c) + u_\eta(c) = c(u_\eta - u_\xi)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \left(\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\right)[c(u_\eta - u_\xi)] = \left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)[c(u_\eta - u_\xi)] = c^2(u_{\eta\eta} - 2u_{\xi\eta} + u_{\xi\xi})$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi(1) + u_\eta(1) = u_\eta + u_\xi$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)(u_\eta + u_\xi) = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)(u_\eta + u_\xi) = u_{\eta\eta} + 2u_{\xi\eta} + u_{\xi\xi}$$

Substitute these formulas into the PDE and simplify.

$$c^2(u_{\eta\eta} - 2u_{\xi\eta} + u_{\xi\xi}) - c^2(u_{\eta\eta} + 2u_{\xi\eta} + u_{\xi\xi}) = 0$$
$$-4c^2 u_{\xi\eta} = 0$$
$$u_{\xi\eta} = 0$$

This is known as the first canonical form of the wave equation. Integrate both sides with respect to η.

$$u_\xi = f(\xi)$$

Here f is an arbitrary function of ξ. Integrate both sides with respect to ξ to get u.

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

F and G are arbitrary functions of ξ and η, respectively. Now that u is known, change back to the original variables.

$$u(x,t) = F(x - ct) + G(x + ct)$$

Apply the initial conditions to determine F and G.

$$\begin{aligned} u(x,0) = \log(1+x^2) \quad &\to \quad F(x) + G(x) = \log(1+x^2) \\ u_t(x,0) = 4 + x \quad &\to \quad -cF'(x) + cG'(x) = 4 + x \end{aligned}$$

Multiply both sides of the first equation by c and differentiate both sides with respect to x.

$$\begin{aligned} cF'(x) + cG'(x) &= c\frac{2x}{1+x^2} \\ -cF'(x) + cG'(x) &= 4 + x \end{aligned}$$

Adding the respective sides yields

$$2cG'(x) = c\frac{2x}{1+x^2} + 4 + x,$$

whereas subtracting the respective sides yields

$$2cF'(x) = c\frac{2x}{1+x^2} - 4 - x.$$

As a result,

$$\begin{aligned} F'(x) &= \frac{1}{2}\frac{2x}{1+x^2} - \frac{1}{2c}(4+x) \\ G'(x) &= \frac{1}{2}\frac{2x}{1+x^2} + \frac{1}{2c}(4+x). \end{aligned}$$

Integrate these functions to obtain $F(x)$ and $G(x)$.

$$\begin{aligned} F(x) &= \frac{1}{2}\log(1+x^2) - \frac{1}{4c}(8x + x^2) \\ G(x) &= \frac{1}{2}\log(1+x^2) + \frac{1}{4c}(8x + x^2) \end{aligned}$$

What we solved for are actually $F(w)$ and $G(w)$, where w is any expression we wish.

$$\begin{aligned} F(x-ct) &= \frac{1}{2}\log[1+(x-ct)^2] - \frac{1}{4c}[8(x-ct) + (x-ct)^2] \\ G(x+ct) &= \frac{1}{2}\log[1+(x+ct)^2] + \frac{1}{4c}[8(x+ct) + (x+ct)^2] \end{aligned}$$

Therefore,

$$\begin{aligned} u(x,t) &= F(x-ct) + G(x+ct) \\ &= \frac{1}{2}\log[1+(x-ct)^2] - \frac{1}{4c}[8(x-ct) + (x-ct)^2] + \frac{1}{2}\log[1+(x+ct)^2] + \frac{1}{4c}[8(x+ct) + (x+ct)^2] \\ &= \log\sqrt{[1+(x-ct)^2][1+(x+ct)^2]} + t(4+x). \end{aligned}$$