Exercise 8

A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right) \quad \text{(“spherical wave equation”).}$$

(a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.

(b) Solve for v using (3) and thereby solve the spherical wave equation.

(c) Use (8) to solve it with initial conditions $u(r,0) = \phi(r)$, $u_t(r,0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.

Solution

Part (a)

If we change variables to $v = ru$, we have to write expressions for the derivatives of u in terms of v.

$$\begin{aligned}
v_t &= ru_t \\
v_{tt} &= ru_{tt} \quad \rightarrow \quad \frac{v_{tt}}{r} = u_{tt} \\
v_r &= u + ru_r \\
v_{rr} &= u_r + u_r + ru_{rr} = 2u_r + ru_{rr} \quad \rightarrow \quad \frac{v_{rr}}{r} = u_{rr} + \frac{2}{r}u_r
\end{aligned}$$

As a result of making the change of variables, the PDE becomes

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right) \quad \rightarrow \quad \frac{v_{tt}}{r} = c^2\left(\frac{v_{rr}}{r}\right).$$

Therefore,

$$v_{tt} = c^2 v_{rr}.$$

Part (b)

Equation (3) in the book tells us that the solution to the one-dimensional wave equation above is

$$v(r,t) = f(r-ct) + g(r+ct), \tag{3}$$

where f and g are arbitrary functions. Now that we know v, we can solve for u.

$$ru = f(r-ct) + g(r+ct)$$

Therefore, the solution to the spherical wave equation is

$$u(r,t) = \frac{f(r-ct) + g(r+ct)}{r}.$$

Part (c)

The unknown functions, f and g, are determined by using the initial conditions.

$$\begin{aligned} u(r,0) = \phi(r) \quad &\rightarrow \quad \frac{1}{r}[f(r) + g(r)] = \phi(r) \\ u_t(r,0) = \psi(r) \quad &\rightarrow \quad \frac{1}{r}[-cf'(r) + cg'(r)] = \psi(r) \end{aligned}$$

Multiply both sides of the first equation by cr and both sides of the second equation by r.

$$\begin{aligned} cf(r) + cg(r) &= cr\phi(r) \\ -cf'(r) + cg'(r) &= r\psi(r) \end{aligned}$$

Differentiate both sides of the first equation with respect to r.

$$cf'(r) + cg'(r) = c[r\phi(r)]' \tag{1}$$
$$-cf'(r) + cg'(r) = r\psi(r) \tag{2}$$

Subtract both sides of equation (2) from those of equation (1).

$$2cf'(r) = c[r\phi(r)]' - r\psi(r).$$

Add both sides of equation (2) to those of equation (1).

$$2cg'(r) = c[r\phi(r)]' + r\psi(r).$$

Solve the previous two equations for f' and g'.

$$\begin{aligned} f'(r) &= \frac{1}{2}[r\phi(r)]' - \frac{1}{2c}r\psi(r) \\ g'(r) &= \frac{1}{2}[r\phi(r)]' + \frac{1}{2c}r\psi(r) \end{aligned}$$

Now integrate both sides of each equation to find f and g.

$$\begin{aligned} f(r) &= \frac{1}{2}[r\phi(r)] - \frac{1}{2c}\int^r s\psi(s)\,ds \\ g(r) &= \frac{1}{2}[r\phi(r)] + \frac{1}{2c}\int^r s\psi(s)\,ds \end{aligned}$$

What we solved for are actually $f(w)$ and $g(w)$, where w is any expression we wish.

$$\begin{aligned} f(r-ct) &= \frac{1}{2}[(r-ct)\phi(r-ct)] - \frac{1}{2c}\int^{r-ct} s\psi(s)\,ds \\ g(r+ct) &= \frac{1}{2}[(r+ct)\phi(r+ct)] + \frac{1}{2c}\int^{r+ct} s\psi(s)\,ds. \end{aligned}$$

Plug these formulas into $u(r,t)$ to obtain the solution to the initial value problem.

$$\begin{aligned}
u(r,t) &= \frac{1}{r}[f(r-ct)+g(r+ct)] \\
&= \frac{1}{r}\left\{\frac{1}{2}[(r-ct)\phi(r-ct)] - \frac{1}{2c}\int^{r-ct} s\psi(s)\,ds + \frac{1}{2}[(r+ct)\phi(r+ct)] + \frac{1}{2c}\int^{r+ct} s\psi(s)\,ds\right\} \\
&= \frac{1}{2r}\left[(r-ct)\phi(r-ct)+(r+ct)\phi(r+ct)\right] + \frac{1}{2cr}\left[\int^{r+ct} s\psi(s)\,ds - \int^{r-ct} s\psi(s)\,ds\right] \\
&= \frac{1}{2r}\left[(r-ct)\phi(r-ct)+(r+ct)\phi(r+ct)\right] + \frac{1}{2cr}\left[\int^{r+ct} s\psi(s)\,ds + \int_{r-ct} s\psi(s)\,ds\right]
\end{aligned}$$

Therefore,

$$u(r,t) = \frac{1}{2r}\left[(r-ct)\phi(r-ct)+(r+ct)\phi(r+ct)\right] + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\psi(s)\,ds.$$