Exercise 9

Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0,\ u(x,0) = x^2,\ u_t(x,0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)

Solution

Solution by Operator Factorization

To solve this PDE on the whole line ($-\infty < x < \infty$), the method of operator factorization can be used.

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x \partial t} - 4\frac{\partial^2 u}{\partial t^2} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} - 3\frac{\partial^2}{\partial x \partial t} - 4\frac{\partial^2}{\partial t^2}\right) u = 0$$

$$\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) u = 0$$

If we let

$$v = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) u,$$

then the previous equation becomes

$$\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right) v = 0.$$

In other words, the method of operator factorization reduces the second-order PDE to a system of first-order PDEs.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = v \\ \dfrac{\partial v}{\partial t} - \dfrac{1}{4}\dfrac{\partial v}{\partial x} = 0 \end{cases}$$

Note that the differential of a two-dimensional function $h(x,t)$ is defined as

$$dh = \frac{\partial h}{\partial t}\,dt + \frac{\partial h}{\partial x}\,dx.$$

Dividing both sides by dt results in the fundamental relationship between the total derivative of h and its partial derivatives.

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + \frac{dx}{dt}\frac{\partial h}{\partial x}$$

Along the curves in the tx-plane defined by

$$\frac{dx}{dt} = -\frac{1}{4}, \quad x(\xi, 0) = \xi, \tag{1}$$

where ξ is a characteristic coordinate, the PDE for v becomes an ODE.

$$\frac{dv}{dt} = 0 \tag{2}$$

Integrate both sides of equation (2) with respect to t.

$$v(\xi, t) = f(\xi)$$

Here f is an arbitrary function of the characteristic coordinate. Now integrate both sides of equation (1) with respect to t.

$$x = -\frac{1}{4}t + \xi \quad \rightarrow \quad \xi = x + \frac{1}{4}t$$

In terms of the original variables, then,

$$v(x, t) = f\left(x + \frac{1}{4}t\right).$$

Consequently, the PDE for u is

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = f\left(x + \frac{1}{4}t\right).$$

Along the curves in the tx-plane defined by

$$\frac{dx}{dt} = 1, \quad x(\eta, 0) = \eta, \tag{3}$$

where η is another characteristic coordinate, the PDE for u becomes an ODE.

$$\frac{du}{dt} = f\left(x + \frac{1}{4}t\right) \tag{4}$$

Integrate both sides of equation (3) with respect to t.

$$x = t + \eta \quad \rightarrow \quad \eta = x - t$$

Eliminate x from equation (4).

$$\frac{du}{dt} = f\left(\eta + \frac{5}{4}t\right)$$

Integrate both sides with respect to t.

$$u(\eta, t) = F\left(\eta + \frac{5}{4}t\right) + G(\eta)$$

Here F and G are arbitrary functions. Now that u is known, change back to the original variables to obtain the general solution.

$$u(x, t) = F\left(x + \frac{1}{4}t\right) + G(x - t)$$

Apply the initial conditions to determine F and G.

$$\begin{aligned} u(x, 0) = x^2 \quad &\rightarrow \quad F(x) + G(x) = x^2 \\ u_t(x, 0) = e^x \quad &\rightarrow \quad \frac{1}{4}F'(x) - G'(x) = e^x \end{aligned}$$

Differentiate both sides of the first equation with respect to x.

$$\begin{aligned} F'(x) + G'(x) &= 2x \\ \frac{1}{4}F'(x) - G'(x) &= e^x \end{aligned}$$

Solve this system of equations for $F'(x)$ and $G'(x)$.

$$F'(x) = \frac{8}{5}x + \frac{4}{5}e^x$$

$$G'(x) = \frac{2}{5}x - \frac{4}{5}e^x$$

Integrate these functions to obtain $F(x)$ and $G(x)$.

$$F(x) = \frac{4}{5}x^2 + \frac{4}{5}e^x$$

$$G(x) = \frac{1}{5}x^2 - \frac{4}{5}e^x$$

What we solved for are actually $F(w)$ and $G(w)$, where w is any expression we wish.

$$F\left(x + \frac{1}{4}t\right) = \frac{4}{5}\left(x + \frac{1}{4}t\right)^2 + \frac{4}{5}e^{x+t/4}$$

$$G(x-t) = \frac{1}{5}(x-t)^2 - \frac{4}{5}e^{x-t}$$

Therefore,

$$\begin{aligned} u(x,t) &= F\left(x + \frac{1}{4}t\right) + G(x-t) \\ &= \frac{4}{5}\left(x + \frac{1}{4}t\right)^2 + \frac{4}{5}e^{x+t/4} + \frac{1}{5}(x-t)^2 - \frac{4}{5}e^{x-t} \\ &= x^2 + \frac{1}{4}t^2 + \frac{4}{5}(e^{x+t/4} - e^{x-t}) \\ &= x^2 + \frac{1}{4}t^2 + \frac{4}{5}e^{x-t}(e^{5t/4} - 1). \end{aligned}$$

Solution by the Method of Characteristics

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0$$

Comparing this equation with the general form of a second-order PDE,
$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G$, we see that $A = -4$, $B = -3$, $C = 1$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dx}{dt} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$

$$\frac{dx}{dt} = \frac{1}{-8}\left(-3 \pm \sqrt{9+16}\right)$$

$$\frac{dx}{dt} = -\frac{1}{4} \quad \text{or} \quad \frac{dx}{dt} = 1.$$

Note that the discriminant, $B^2 - 4AC = 25$, is greater than 0, which means that the PDE is hyperbolic. That means the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the tx-plane.

$$x = -\frac{1}{4}t + C_1 \quad \text{or} \quad x = t + C_2.$$

Solve for the constants of integration.

$$C_1 = x + \frac{1}{4}t$$

$$C_2 = x - t$$

Make the change of variables, $\xi = x + t/4$ and $\eta = x - t$, so that the PDE takes the simplest form. Use the chain rule to determine the old derivatives in terms of these new variables.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = u_\xi\left(\frac{1}{4}\right) + u_\eta(-1) = \frac{1}{4}u_\xi - u_\eta$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \left(\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\right)\left(\frac{1}{4}u_\xi - u_\eta\right) = \left(\frac{1}{4}\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right)\left(\frac{1}{4}u_\xi - u_\eta\right) = \frac{1}{16}u_{\xi\xi} - \frac{1}{2}u_{\xi\eta} + u_{\eta\eta}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi(1) + u_\eta(1) = u_\xi + u_\eta$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)(u_\xi + u_\eta) = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)(u_\xi + u_\eta) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)\left(\frac{1}{4}u_\xi - u_\eta\right) = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{1}{4}u_\xi - u_\eta\right) = \frac{1}{4}u_{\xi\xi} - \frac{3}{4}u_{\xi\eta} - u_{\eta\eta}$$

Substitute these formulas into the PDE and simplify.

$$(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) - 3\left(\frac{1}{4}u_{\xi\xi} - \frac{3}{4}u_{\xi\eta} - u_{\eta\eta}\right) - 4\left(\frac{1}{16}u_{\xi\xi} - \frac{1}{2}u_{\xi\eta} + u_{\eta\eta}\right) = 0$$

$$\frac{25}{4}u_{\xi\eta} = 0$$

Multiply both sides by 4/25.

$$u_{\xi\eta} = 0$$

This is known as the first canonical form of the PDE. Integrate both sides with respect to η.

$$u_\xi = f(\xi)$$

Here f is an arbitrary function of ξ. Integrate both sides with respect to ξ to get u.

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

F and G are arbitrary functions of ξ and η, respectively. Now that u is known, change back to the original variables.

$$u(x,t) = F\left(x + \frac{1}{4}t\right) + G(x - t)$$

Apply the initial conditions to determine F and G.

$$\begin{aligned} u(x,0) = x^2 \quad &\rightarrow \quad F(x) + G(x) = x^2 \\ u_t(x,0) = e^x \quad &\rightarrow \quad \frac{1}{4}F'(x) - G'(x) = e^x \end{aligned}$$

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Therefore,

$$\begin{aligned} u(x,t) = F\left(x + \frac{1}{4}t\right) + G(x - t) &= \frac{4}{5}\left(x + \frac{1}{4}t\right)^2 + \frac{4}{5}e^{x+t/4} + \frac{1}{5}(x - t)^2 - \frac{4}{5}e^{x-t} \\ &= x^2 + \frac{1}{4}t^2 + \frac{4}{5}e^{x-t}(e^{5t/4} - 1). \end{aligned}$$