# Exercise 1

Use the energy conservation of the wave equation to prove that the only solution with $\phi \equiv 0$ and $\psi \equiv 0$ is $u \equiv 0$. (*Hint:* Use the first vanishing theorem in Section A.1.)

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### Solution

Following the hint, we will use the first vanishing theorem. This theorem states the following: Let $f(x)$ be a continuous function in a finite closed interval $[a, b]$. Assume that $f(x) \geq 0$ in the interval and that $\int_a^b f(x)\,dx = 0$. Then $f(x) = 0$. We will also use the law of conservation of energy for the wave equation,

$$E = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\,dx.$$

Even though this integral goes from $-\infty$ to $\infty$, we can use the vanishing theorem as long as $\phi(x)$ and $\psi(x)$ vanish outside an interval, $|x| \leq R$. The reason is that due to causality, $u(x, t)$ and hence its derivatives vanish for $|x| \geq R + ct$. Because $\phi(x) = 0$ and $\psi(x) = 0$ initially, there is no energy at $t = 0$. The energy is constant in time, so $E = 0$ for all time.

$$0 = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\,dx$$

As a result of the first vanishing theorem,

$$\rho u_t^2 + T u_x^2 = 0$$

$$\frac{\rho}{T} u_t^2 = -u_x^2.$$

Since $\rho$, $T$, $u_x^2$, and $u_t^2$ are positive, the only way this equation can be satisfied is if $u_t^2 = u_x^2 = 0$. That is,

$$\begin{aligned} u_t = 0 \quad &\rightarrow \quad u = f(x) \\ u_x = 0 \quad &\rightarrow \quad u = g(t), \end{aligned}$$

where $f$ and $g$ are arbitrary functions. Hence,

$$u = f(x) = g(t).$$

The only way a function of $x$ can be equal to a function of $t$ is if the function in question is a constant. And since $\phi(x)$ and $\psi(x)$ are equal to zero, this constant must be zero. Therefore, $u = 0$.