# Exercise 4

If $u(x,t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity

$$u(x+h,\, t+k) + u(x-h,\, t-k) = u(x+k,\, t+h) + u(x-k,\, t-h)$$

for all $x$, $t$, $h$, and $k$. Sketch the quadrilateral $Q$ whose vertices are the arguments in the identity.

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### Solution

It is important to recognize here that the points, $P_1 = (x+h,\, t+k)$, $P_2 = (x-h,\, t-k)$, $Q_1 = (x+k,\, t+h)$, and $Q_2 = (x-k,\, t-h)$, are the vertices of a quadrilateral, specifically a parallelogram. $P_1$ is diagonally opposite to $P_2$, and $Q_1$ is diagonally opposite to $Q_2$. Shown below to illustrate this is an example with $h = 1$ and $k = 2$ and $x = t = 0$.

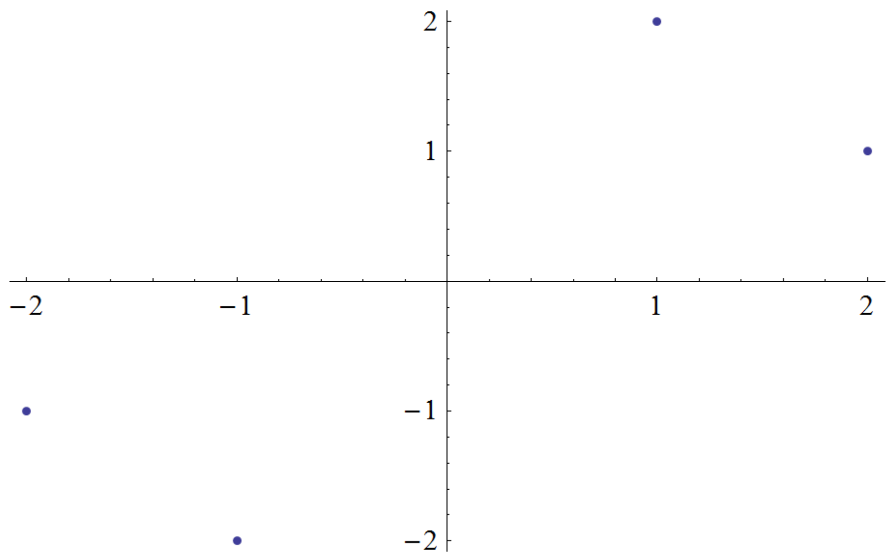


Figure 1: A plot of the four points, $(1,2)$, $(-1,-2)$, $(2,1)$, and $(-2,-1)$.

Define the two functions, $A(x,t)$ and $B(x,t)$, as follows.

$$A(x,t) = \frac{1}{2}\left[f(x-t) - \int^{x-t} g(s)\,ds\right]$$
$$B(x,t) = \frac{1}{2}\left[f(x+t) + \int^{x+t} g(s)\,ds\right]$$

The point of this is so that when we add them, we get D'Alembert's solution for the wave equation for all $x$ and $t$.

$$u(x,t) = A(x,t) + B(x,t)$$

Note that $A(x,t)$ is constant along the characteristic lines, $\xi = x - t$, and that $B(x,t)$ is constant along the characteristic lines, $\eta = x + t$. What this means is that $A(x,t)$ evaluated at every point along a characteristic line has the same value, so because $P_1$ and $Q_1$ are both along an $x - t$ characteristic, $A(P_1)$ and $A(Q_1)$ are equal. Similarly, $P_1$ and $Q_2$ are both points along an $x + t$ characteristic, so $B(P_1)$ and $B(Q_2)$ are equal.

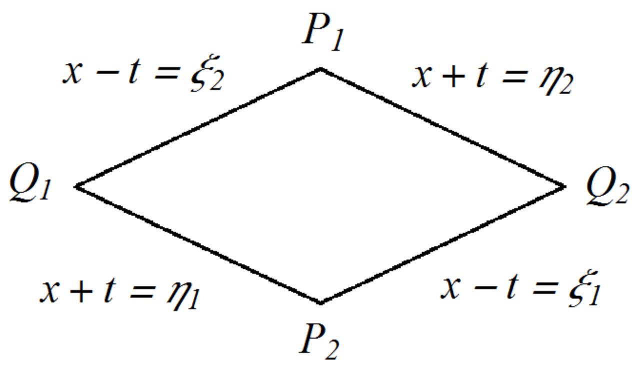


Figure 2: Schematic of the parallelogram formed by the characteristics.

Examining Figure 2, the following relationships can be made.

$$\begin{aligned} A(P_1) &= A(Q_1) \quad \text{and} \quad A(P_2) = A(Q_2) \\ B(P_1) &= B(Q_2) \quad \text{and} \quad B(P_2) = B(Q_1), \end{aligned}$$

The left side of the equation we are trying to prove can be written as $u(P_1) + u(P_2)$. With the relationships in hand, we can infer the right side.

$$u(P_1) + u(P_2) = [A(P_1) + B(P_1)] + [A(P_2) + B(P_2)]$$

Apply the relationships here.

$$u(P_1) + u(P_2) = [A(Q_1) + B(Q_2)] + [A(Q_2) + B(Q_1)]$$

Switch terms around.

$$u(P_1) + u(P_2) = [A(Q_1) + B(Q_1)] + [A(Q_2) + B(Q_2)]$$

And since the right side is $u(Q_1) + u(Q_2)$, we obtain the desired result.

$$u(P_1) + u(P_2) = u(Q_1) + u(Q_2)$$

Therefore,

$$u(x + h, t + k) + u(x - h, t - k) = u(x + k, t + h) + u(x - k, t - h)$$

for all $x$, $t$, $h$, and $k$.