Exercise 6

Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in n-dimensional space satisfies the PDE

$$u_{tt} = c^2\left(u_{rr} + \frac{n-1}{r}u_r\right),$$

where r is the spherical coordinate. Consider such a wave that has the special form $u(r,t) = \alpha(r)f(r - \beta(r))$, where $\alpha(r)$ is called the attenuation and $\beta(r)$ the delay. The question is whether such solutions exist for "arbitrary" functions f.

(a) Plug the special form into the PDE to get an ODE for f.

(b) Set the coefficients of f'', f', and f equal to zero.

(c) Solve the ODEs to see that $n = 1$ or $n = 3$ (unless $u \equiv 0$).

(d) If $n = 1$, show that $\alpha(r)$ is a constant (so that "there is no attenuation").

(T. Morley, *American Mathematical Monthly*, Vol. 27, pp. 69–71, 1985)

Solution

Part (a)

Evaluate the derivatives of the special form. Once we find u_{tt}, u_{rr}, and u_r, we'll be able to get the ODE.

$$\begin{aligned}
u_t &= \alpha f' \\
u_{tt} &= \alpha f'' \\
u_r &= \alpha' f + \alpha(-\beta')f' = \alpha' f - \alpha\beta' f' \\
u_{rr} &= \alpha'' f + \alpha'(-\beta')f' + \alpha'(-\beta')f' + \alpha(-\beta')^2 f'' + \alpha(-\beta'')f'
\end{aligned}$$

Simplifying u_{rr} gives us

$$u_{rr} = \alpha'' f - (2\alpha'\beta' + \alpha\beta'')f' + \alpha\beta'^2 f''.$$

Now we can substitute these expressions for the terms in the PDE to get the ODE for f.

$$u_{tt} = c^2\left(u_{rr} + \frac{n-1}{r}u_r\right)$$

$$\alpha f'' = c^2\left[\alpha'' f - (2\alpha'\beta' + \alpha\beta'')f' + \alpha\beta'^2 f'' + \frac{n-1}{r}(\alpha' f - \alpha\beta' f')\right]$$

Simplifying this gives us the following.

$$0 = c^2\left(\alpha'' + \frac{n-1}{r}\alpha'\right)f - c^2\left(2\alpha'\beta' + \alpha\beta'' + \frac{n-1}{r}\alpha\beta'\right)f' + \alpha(c^2\beta'^2 - 1)f''$$

Part (b)

Setting the coefficients equal to zero yields a system of differential equations.

$$\alpha'' + \frac{n-1}{r}\alpha' = 0 \tag{1}$$

$$2\alpha'\beta' + \alpha\beta'' + \frac{n-1}{r}\alpha\beta' = 0 \tag{2}$$

$$c^2\beta'^2 - 1 = 0 \tag{3}$$

Part (c)

If we make the substitution, $p = \alpha'$, (1) will be first-order, and we'll be able to solve it with an integrating factor.

$$p' + \frac{n-1}{r}p = 0$$

Multiply both sides by the integrating factor,

$$I(r) = e^{\int^r \frac{n-1}{s}\,ds} = e^{(n-1)\ln r} = r^{n-1}.$$

$$r^{n-1}p' + (n-1)r^{n-2}p = 0$$

The left side is just the derivative of the product, $r^{n-1}p$.

$$(r^{n-1}p)' = 0$$

Integrate both sides.

$$r^{n-1}p = C_1$$

Now that we solved for p, substitute it for α'.

$$\alpha' = C_1 r^{1-n}$$

Integrate this result to get α.

$$\alpha(r) = \frac{C_1}{2-n}r^{2-n} + C_2$$

(3) can be solved quite easily for β. Isolate β'^2, take the square root of both sides, and integrate to get the answer.

$$\beta'^2 = \frac{1}{c^2}$$

$$\beta' = \pm\frac{1}{c}$$

$$\beta(r) = \pm\frac{1}{c}r + C_3$$

Now that we know α and β, we can substitute these results into (2). Note that $\beta'' = 0$.

$$2C_1 r^{1-n}\left(\pm\frac{1}{c}\right) + \frac{n-1}{r}\left(\frac{C_1}{2-n}r^{2-n} + C_2\right)\left(\pm\frac{1}{c}\right) = 0$$

Simplifying this gives us

$$C_1 r^{1-n}\left(\frac{3-n}{2-n}\right) + \frac{n-1}{r}C_2 = 0. \tag{4}$$

The only values of n that give us solutions are $n = 1$ and $n = 3$. Anything else would require C_1 and C_2 to be 0, which leads only to the trivial solution, $u = 0$.

Part (d)

If $n = 1$, then (4) becomes

$$2C_1 = 0,$$

which implies that C_1 equals zero. Plugging $C_1 = 0$ into the equation for α gives $\alpha(r) = C_2$. Therefore, $\alpha(r)$ is a constant and there is no attenuation of the n-dimensional spherical wave.