Exercise 7

(a) More generally, if $u_t - ku_{xx} = f$, $v_t - kv_{xx} = g$, $f \leq g$, and $u \leq v$ at $x = 0$, $x = l$ and $t = 0$, prove that $u \leq v$ for $0 \leq x \leq l$, $0 \leq t < \infty$.

(b) If $v_t - v_{xx} \geq \sin x$ for $0 \leq x \leq \pi$, $0 < t < \infty$, and if $v(0,t) \geq 0$, $v(\pi,t) \geq 0$ and $v(x,0) \geq \sin x$, use part (a) to show that $v(x,t) \geq (1 - e^{-t}) \sin x$.

Solution

Part (a)

The two given equations are inhomogeneous diffusion equations.

$$v_t - kv_{xx} = g \tag{1}$$
$$u_t - ku_{xx} = f \tag{2}$$

We're told that $v \geq u$ initially and on the boundaries. That is,

$$\begin{aligned} v(x,0) \geq u(x,0) \quad &\rightarrow \quad v(x,0) - u(x,0) \geq 0 \\ v(0,t) \geq u(0,t) \quad &\rightarrow \quad v(0,t) - u(0,t) \geq 0 \\ v(l,t) \geq u(l,t) \quad &\rightarrow \quad v(l,t) - u(l,t) \geq 0. \end{aligned}$$

Consider the difference between (1) and (2).

$$v_t - kv_{xx} - (u_t - ku_{xx}) = g - f$$

$$v_t - u_t - kv_{xx} + ku_{xx} = g - f$$

Now factor the equation.

$$(v - u)_t - k(v - u)_{xx} = g - f$$

Let $w = v - u$ so we get

$$w_t - kw_{xx} = g - f, \quad w(x,0) \geq 0, \; w(0,t) \geq 0, \; w(l,t) \geq 0.$$

And because $g \geq f$,

$$w_t - kw_{xx} \geq 0.$$

To look for the smallest w, consider only

$$w_t - kw_{xx} = 0.$$

According to the minimum principle, w will have a minimum either initially or on the boundary. Because the lowest value of w on each of the conditions is 0, $w \geq 0$ for $0 \leq x \leq l$ and $0 \leq t < \infty$. Therefore, $v \geq u$ for $0 \leq x \leq l$ and $0 \leq t < \infty$.

Part (b)

Let $u(x,t) = (1 - e^{-t}) \sin x$. If we can show that all the conditions of part (a) hold, then the conclusion that $v \geq u$ must be true. Take derivatives of u and substitute it into the diffusion equation.

$$\begin{aligned} u_t &= e^{-t} \sin x \\ u_x &= (1 - e^{-t}) \cos x \\ u_{xx} &= -(1 - e^{-t}) \sin x \end{aligned}$$

Hence, $u_t - u_{xx} = \sin x$. Because we're told that $v_t - v_{xx} \geq \sin x$, the first condition of part (a) holds. Let's check the initial and boundary conditions now.

$$\begin{aligned} u(0,t) &= 0 \\ u(\pi,t) &= 0 \\ u(x,0) &= 0 \end{aligned}$$

Since $v(0,t) \geq 0$, $v(\pi,t) \geq 0$, and $v(x,0) \geq 0$ from $x = 0$ to $x = \pi$, the rest of the conditions in part (a) are satisfied. Therefore,

$$v(x,t) \geq (1 - e^{-t}) \sin x$$

for $0 \leq x \leq \pi$ and $0 \leq t < \infty$.