Exercise 10

(a) Solve Exercise 9 using the general formula discussed in the text. This expresses $u(x,t)$ as a certain integral. Substitute $p = (x-y)/\sqrt{4kt}$ in this integral.

(b) Since this solution is unique, the resulting formula must agree with the answer to Exercise 9. Deduce the value of
$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp.$$

Solution

Part (a)

The general solution to the initial value problem in Exercise 9,

$$u_t = ku_{xx}, \quad u(x,0) = \phi(x) = x^2,$$

is given by
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\,dy.$$

Plug $\phi(x) = x^2$ into the formula.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} y^2 e^{-\frac{(x-y)^2}{4kt}}\,dy.$$

Now we make the recommended substitution in the problem statement.

$$p = \frac{x-y}{\sqrt{4kt}} \quad \rightarrow \quad y = x - \sqrt{4kt}p$$
$$dp = -\frac{dy}{\sqrt{4kt}}$$

Part (b)

The integral becomes
$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_{\infty}^{-\infty} (x - \sqrt{4kt}p)^2 e^{-p^2}\,dp.$$

Expand the polynomial in parentheses and switch the limits of integration to eliminate the minus sign.
$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x^2 - 2\sqrt{4kt}xp + 4ktp^2) e^{-p^2}\,dp$$

Split the integral up and bring the constants out in front.

$$u(x,t) = \frac{1}{\sqrt{\pi}} \left[x^2 \int_{-\infty}^{\infty} e^{-p^2}\,dp - 2\sqrt{4kt}x \int_{-\infty}^{\infty} pe^{-p^2}\,dp + 4kt \int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp \right]$$

The first integral evaluates to $\sqrt{\pi}$, and the second integral evaluates to 0 because the integrand is a product of an odd function p and an even function e^{-p^2} and it is being integrated over a symmetric interval.
$$u(x,t) = \frac{1}{\sqrt{\pi}} \left[x^2 \cdot \sqrt{\pi} - 0 + 4kt \int_{-\infty}^{\infty} p^2 e^{-p^2}\,dp \right]$$

Hence,

$$u(x,t) = x^2 + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp.$$

Since the solution is unique, this expression for $u(x,t)$ must be equal to the solution we derived in Exercise 9, $u(x,t) = x^2 + 2kt$, for the same initial value problem.

$$\cancel{x^2} + \frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp = \cancel{x^2} + 2kt$$

Cancel the common terms from both sides.

$$\frac{4kt}{\sqrt{\pi}} \int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp = 2kt$$

Therefore,

$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp = \frac{\sqrt{\pi}}{2}.$$