Exercise 11

(a) Consider the diffusion equation on the whole line with the usual initial condition $u(x, 0) = \phi(x)$. If $\phi(x)$ is an *odd* function, show that the solution $u(x, t)$ is also an *odd* function of x. (*Hint:* Consider $u(-x, t) + u(x, t)$ and use the uniqueness.)

(b) Show that the same is true if "odd" is replaced by "even."

(c) Show that the analogous statements are true for the wave equation.

Solution

Part (a)

The initial value problem to consider here is

$$u_t = ku_{xx}, \quad u(x, 0) = \phi(x).$$

Because $\phi(x)$ is an odd function, it implies that $\phi(-x) = -\phi(x)$. The associated initial value problem for $u(-x, t)$ is

$$u_t(-x, t) = ku_{xx}(-x, t), \quad u(-x, 0) = \phi(-x) = -\phi(x).$$

Following the hint, we will consider $u(-x, t) + u(x, t)$. Since $u(-x, t)$ and $u(x, t)$ both satisfy the diffusion equation, a linear combination of the two is also a solution to the equation.

$$[u(-x, t) + u(x, t)]_t = k[u(-x, t) + u(x, t)]_{xx}, \quad u(-x, 0) + u(x, 0) = -\phi(x) + \phi(x) = 0$$

A solution to this equation that satisfies the initial condition is $u(-x, t) + u(x, t) = 0$, and because the solution to the diffusion equation is unique, it is the one and only solution. Bring $u(x, t)$ to the right side to get the final result.

$$u(-x, t) = -u(x, t)$$

Therefore, if $\phi(x)$ is an odd function, then $u(x, t)$ is an odd function of x.

Part (b)

The initial value problem to consider here is

$$u_t = ku_{xx}, \quad u(x, 0) = \phi(x).$$

Because $\phi(x)$ is an even function, it implies that $\phi(-x) = \phi(x)$. The associated initial value problem for $u(-x, t)$ is

$$u_t(-x, t) = ku_{xx}(-x, t), \quad u(-x, 0) = \phi(-x) = \phi(x).$$

Here we will consider $u(-x, t) - u(x, t)$. Since $u(-x, t)$ and $u(x, t)$ both satisfy the diffusion equation, a linear combination of the two is also a solution to the equation.

$$[u(-x, t) - u(x, t)]_t = k[u(-x, t) - u(x, t)]_{xx}, \quad u(-x, 0) - u(x, 0) = \phi(x) - \phi(x) = 0$$

A solution to this equation that satisfies the initial condition is $u(-x,t) - u(x,t) = 0$, and because the solution to the diffusion equation is unique, it is the one and only solution. Bring $u(x,t)$ to the right side to get the final result.

$$u(-x,t) = u(x,t)$$

Therefore, if $\phi(x)$ is an even function, then $u(x,t)$ is an even function of x.

Part (c)

The wave equation has a term with two time derivatives, so there will be two initial conditions, $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$. We will show that if $\phi(x)$ and $\psi(x)$ are odd, then $u(x,t)$ is an odd function of x.

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = \phi(x),\ u_t(x,0) = \psi(x).$$

Because $\phi(x)$ and $\psi(x)$ are odd functions, it implies that $\phi(-x) = -\phi(x)$ and $\psi(-x) = -\psi(x)$. The associated initial value problem for $u(-x,t)$ is

$$u_{tt}(-x,t) = c^2 u_{xx}(-x,t), \quad u(-x,0) = \phi(-x) = -\phi(x),\ u_t(-x,0) = \psi(-x) = -\psi(x).$$

Here we will consider $u(-x,t) + u(x,t)$. Since $u(-x,t)$ and $u(x,t)$ both satisfy the wave equation, a linear combination of the two is also a solution to the equation.

$$\begin{aligned} [u(-x,t) + u(x,t)]_{tt} &= c^2[u(-x,t) + u(x,t)]_{xx} \\ u(-x,0) + u(x,0) &= -\phi(x) + \phi(x) = 0 \\ u_t(-x,0) + u_t(x,0) &= -\psi(x) + \psi(x) = 0 \end{aligned}$$

A solution to this equation that satisfies the initial conditions is $u(-x,t) + u(x,t) = 0$, and because the solution to the wave equation is unique, it is the one and only solution. Bring $u(x,t)$ to the right side to get the final result.

$$u(-x,t) = -u(x,t)$$

Therefore, if $\phi(x)$ and $\psi(x)$ are odd functions, then $u(x,t)$ is an odd function of x.

Now we will show that if $\phi(x)$ and $\psi(x)$ are even functions, that is, $\phi(-x) = \phi(x)$ and $\psi(-x) = \psi(x)$, then $u(x,t)$ is an even function of x. The equations satisfied by $u(x,t)$ and $u(-x,t)$ are the following.

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & u(x,0) &= \phi(x),\ u_t(x,0) = \psi(x) \\ u_{tt}(-x,t) &= c^2 u_{xx}(-x,t), & u(-x,0) &= \phi(-x) = \phi(x),\ u_t(-x,0) = \psi(-x) = \psi(x) \end{aligned}$$

Here we will consider $u(-x,t) - u(x,t)$. Since $u(-x,t)$ and $u(x,t)$ both satisfy the wave equation, a linear combination of the two is also a solution to the equation.

$$\begin{aligned} [u(-x,t) - u(x,t)]_{tt} &= c^2[u(-x,t) - u(x,t)]_{xx} \\ u(-x,0) - u(x,0) &= \phi(x) - \phi(x) = 0 \\ u_t(-x,0) - u_t(x,0) &= \psi(x) - \psi(x) = 0 \end{aligned}$$

A solution to this equation that satisfies the initial conditions is $u(-x,t) - u(x,t) = 0$, and because the solution to the wave equation is unique, it is the one and only solution. Bring $u(x,t)$ to the right side to get the final result.

$$u(-x,t) = u(x,t)$$

Therefore, if $\phi(x)$ and $\psi(x)$ are even functions, then $u(x,t)$ is an even function of x.