Exercise 12

The purpose of this exercise is to calculate $Q(x,t)$ approximately for large t. Recall that $Q(x,t)$ is the temperature of an infinite rod that is initially at temperature 1 for $x > 0$, and 0 for $x < 0$.

(a) Express $Q(x,t)$ in terms of $\mathscr{E}\text{rf}$.

(b) Find the Taylor series of $\mathscr{E}\text{rf}(x)$ around $x = 0$. (*Hint:* Expand e^z, substitute $z = -y^2$, and integrate term by term.)

(c) Use the first two nonzero terms in this Taylor expansion to find an approximate formula for $Q(x,t)$.

(d) *Why* is this formula a good approximation for x fixed and t large?

Solution

Part (a)

The initial value problem for $Q(x,t)$ is

$$Q_t = kQ_{xx}, \quad Q(x,0) = H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}. \tag{1}$$

To solve (1) we will use the similarity method. Since the Heaviside function $H(x)$ is dimensionless, Q is dimensionless as well. Hence, the variables x (meters), t (seconds), and k (meters^2/second) have to appear in the solution as a dimensionless combination. The combination variable η is thus

$$\frac{x^2}{kt}$$

or any convenient multiple or power thereof. Choose

$$\eta = \frac{x}{\sqrt{4kt}}$$

to make the process of obtaining the final answer smoother. Hence,

$$Q = Q(\eta),$$

Write expressions for Q_t and Q_{xx} in terms of this new variable using the chain rule.

$$\begin{aligned}
\frac{\partial Q}{\partial t} &= \frac{dQ}{d\eta}\frac{\partial \eta}{\partial t} = \frac{dQ}{d\eta}\left(-\frac{x}{4\sqrt{kt^3}}\right) \\
\frac{\partial Q}{\partial x} &= \frac{dQ}{d\eta}\frac{\partial \eta}{\partial x} = \frac{dQ}{d\eta}\left(\frac{1}{\sqrt{4kt}}\right) \\
\frac{\partial^2 Q}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial x}\right) = \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\left(\frac{\partial Q}{\partial x}\right) = \frac{d^2Q}{d\eta^2}\left(\frac{1}{4kt}\right)
\end{aligned}$$

Substituting these expressions into the diffusion equation gives

$$\frac{dQ}{d\eta}\left(-\frac{x}{4\sqrt{kt^3}}\right) = k\frac{d^2Q}{d\eta^2}\left(\frac{1}{4kt}\right).$$

Cancel common terms and bring all terms to one side.

$$Q'' + \frac{x}{\sqrt{kt}} Q' = 0$$

Use the combination variable η.

$$Q'' + 2\eta Q' = 0$$

Make the substitution $r = Q' = dQ/d\eta$.

$$\frac{dr}{d\eta} + 2\eta r = 0$$

This is a first-order differential equation that we can solve by separation of variables.

$$\frac{dr}{r} = -2\eta \, d\eta$$

Integrate both sides.

$$\ln |r| = -\eta^2 + C$$

Exponentiate both sides.

$$|r| = e^{-\eta^2 + C}$$

$$r = \pm e^C e^{-\eta^2}$$

$$r = C_1 e^{-\eta^2}$$

Now that we have r, we can get Q by integrating the result.

$$Q(\eta) = \int_0^\eta r \, ds + C_2 = \int_0^\eta C_1 e^{-s^2} \, ds + C_2$$

The lower limit, 0, is arbitrary; choosing a different value leads to a different value for C_2. To evaluate the constants of integration, we look to the initial condition, $Q(x, 0) = H(x)$. For $x > 0$, as $t \to 0$, $\eta \to +\infty$. Conversely, for $x < 0$, as $t \to 0$, $\eta \to -\infty$. Thus,

$$\text{For } x > 0: \qquad 1 = C_1 \int_0^\infty e^{-s^2} \, ds + C_2$$

$$\text{For } x < 0: \qquad 0 = C_1 \int_0^{-\infty} e^{-s^2} \, ds + C_2$$

The system of equations to solve is

$$1 = C_1 \cdot \frac{\sqrt{\pi}}{2} + C_2$$

$$0 = -C_1 \cdot \frac{\sqrt{\pi}}{2} + C_2.$$

Solving it yields

$$C_1 = \frac{1}{\sqrt{\pi}} \quad \text{and} \quad C_2 = \frac{1}{2}.$$

The solution is thus

$$Q(\eta) = \frac{1}{\sqrt{\pi}} \int_0^\eta e^{-s^2} \, ds + \frac{1}{2} = \frac{1}{2} \operatorname{erf}(\eta) + \frac{1}{2} = \frac{1}{2} [\operatorname{erf}(\eta) + 1].$$

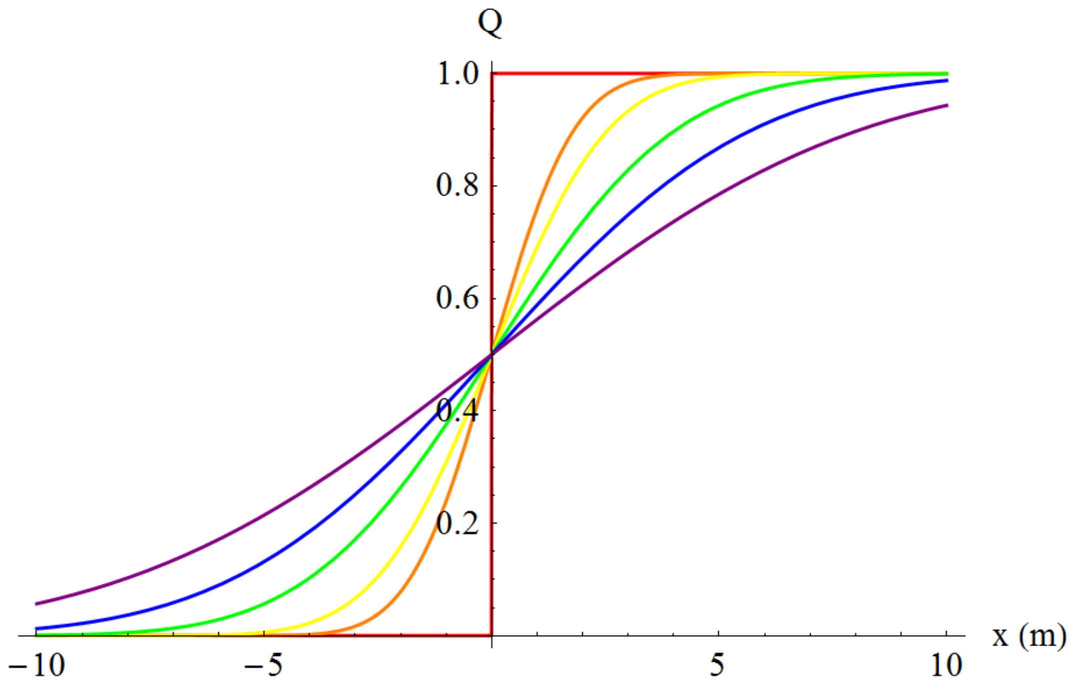

Figure 1: Plot of the dimensionless temperature $Q(x,t)$ with $k = 1$ m^2/s for six different times: $t = 0$ s (red), $t = 1$ s (orange), $t = 2$ s (yellow), $t = 5$ s (green), $t = 10$ s (blue), and $t = 20$ s (purple).

In terms of the original variables it is

$$Q(x,t) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) + 1\right]. \tag{2}$$

Part (b)

Following the hint, we will write the Taylor series of e^z about $z = 0$ (also known as the Maclaurin series), substitute $z = -y^2$, and then integrate the series term by term.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Now make the substitution.

$$e^{-y^2} = \sum_{n=0}^{\infty} \frac{(-y^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{n!}$$

With this series in hand, we can now write the Taylor series expansion of $\operatorname{erf} x$ about $x = 0$. Start with the definition of erf x.

$$\begin{aligned}\operatorname{erf} x &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy \\ &= \frac{2}{\sqrt{\pi}}\int_0^x \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n}}{n!}\,dy \\ &= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^x y^{2n}\,dy \\ &= \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left.\frac{y^{2n+1}}{2n+1}\right|_0^x\end{aligned}$$

Therefore, the Taylor series expansion of erf x centered at $x = 0$ is

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{x^{2n+1}}{2n+1}.$$

Part (c)

The series can be written term by term as follows.

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}x - \frac{2}{3\sqrt{\pi}}x^3 + \frac{1}{5\sqrt{\pi}}x^5 - \frac{1}{21\sqrt{\pi}}x^7 + \cdots$$

If we use only the first two nonzero terms we get an approximation for erf x.

$$\operatorname{erf} x \approx \frac{2}{\sqrt{\pi}}x - \frac{2}{3\sqrt{\pi}}x^3$$

The error function appears with an argument of $x/\sqrt{4kt}$ in the solution for $Q(x,t)$ in (2), so replace x with this expression in the Taylor series.

$$\operatorname{erf}\frac{x}{\sqrt{4kt}} \approx \frac{2}{\sqrt{\pi}}\left(\frac{x}{\sqrt{4kt}}\right) - \frac{2}{3\sqrt{\pi}}\left(\frac{x}{\sqrt{4kt}}\right)^3 = \frac{x}{\sqrt{\pi kt}} - \frac{1}{12}\frac{x^3}{\sqrt{\pi k^3 t^3}}$$

That is,

$$\operatorname{erf}\frac{x}{\sqrt{4kt}} \approx \frac{x}{\sqrt{\pi kt}}\left(1 - \frac{x^2}{12kt}\right).$$

Plugging this two-term form for the error function into (2), we get an approximation for $Q(x,t)$.

$$Q(x,t) \approx \frac{1}{2}\left[\frac{x}{\sqrt{\pi kt}}\left(1 - \frac{x^2}{12kt}\right) + 1\right]$$

Part (d)

When x is fixed and t is large, the combination variable $\eta = x/\sqrt{4kt} \ll 1$. Therefore, higher-order terms in the Taylor series are negligible compared to the first two. This is why the approximation is a good one.

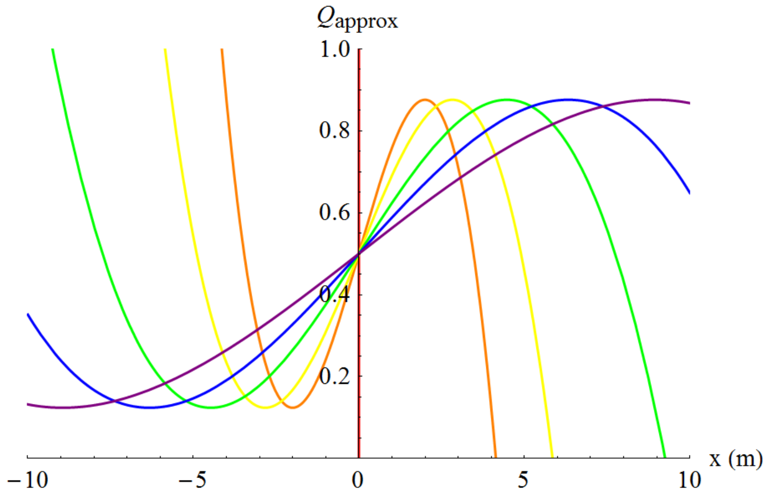

Figure 2: Plot of the two-term approximation to $Q(x,t)$ with $k = 1$ m^2/s for six different times: $t = 0$ s (red), $t = 1$ s (orange), $t = 2$ s (yellow), $t = 5$ s (green), $t = 10$ s (blue), and $t = 20$ s (purple).

If we do a side-by-side comparison with the graph of the exact solution for $Q(x,t)$, we see that as time increases, the interval over which the approximation closely matches the exact solution increases.

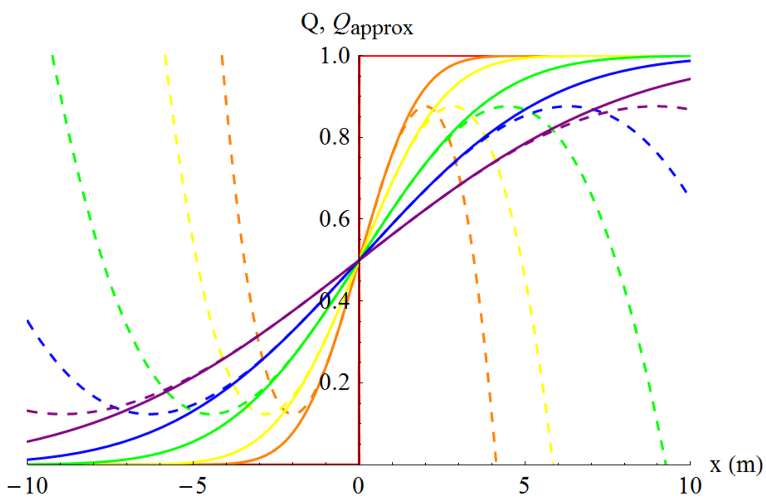

Figure 3: Superposition of the plots of the exact solution for $Q(x,t)$ (solid lines) and the approximate solution for $Q(x,t)$ (dashed lines) with $k = 1$ m^2/s for six different times: $t = 0$ s (red), $t = 1$ s (orange), $t = 2$ s (yellow), $t = 5$ s (green), $t = 10$ s (blue), and $t = 20$ s (purple).