Exercise 17

Solve the diffusion equation with variable dissipation:

$$u_t - ku_{xx} + bt^2 u = 0 \quad \text{for } -\infty < x < \infty \qquad \text{with } u(x,0) = \phi(x),$$

where $b > 0$ is a constant. (*Hint:* The solutions of the ODE $w_t + bt^2 w = 0$ are $Ce^{-bt^3/3}$. So make the change of variables $u(x,t) = e^{-bt^3/3} v(x,t)$ and derive an equation for v.)

Solution

Solution by the Change of Variables

Following the hint, make the change of variables,

$$u(x,t) = e^{-bt^3/3} v(x,t). \tag{1}$$

Write expressions for the derivatives of u in terms of this new variable.

$$\begin{aligned} u_t &= -bt^2 e^{-bt^3/3} v + e^{-bt^3/3} v_t \\ u_x &= e^{-bt^3/3} v_x \\ u_{xx} &= e^{-bt^3/3} v_{xx} \end{aligned}$$

Now substitute these into the PDE.

$$u_t - ku_{xx} + bt^2 u = -\cancel{bt^2 e^{-bt^3/3}} v + e^{-bt^3/3} v_t - k(e^{-bt^3/3} v_{xx}) + \cancel{bt^2 e^{-bt^3/3}} v$$

After cancelling the common terms, what remains is

$$e^{-bt^3/3} v_t - ke^{-bt^3/3} v_{xx} = 0.$$

Multiply both sides by $e^{bt^3/3}$ and bring the second term to the right.

$$v_t = kv_{xx}$$

This is the diffusion equation. Solving (1) for v gives $v(x,t) = e^{bt^3/3} u(x,t)$, so the initial condition is $v(x,0) = e^0 u(x,0) = \phi(x)$. Its solution is given as

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \phi(s)\, ds.$$

Change back to the original variable, $u(x,t)$.

$$e^{bt^3/3} u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \phi(s)\, ds$$

Therefore,

$$u(x,t) = \frac{e^{-bt^3/3}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} \phi(s)\, ds.$$

Solution by the Integrating Factor

Because the PDE is first-order in the derivative with respect to time, we can multiply the PDE by an integrating factor as we would if it were an ODE.

$$u_t + bt^2 u - k u_{xx} = 0$$

The integrating factor is

$$I = e^{\int^t bs^2\,ds} = e^{bt^3/3}.$$

Now multiply both sides of the PDE by it.

$$e^{bt^3/3} u_t + bt^2 e^{bt^3/3} u - k e^{bt^3/3} u_{xx} = 0$$

The first two terms on the left side can be written as the time derivative of the product, $e^{bt^3/3}u$. Also, $e^{bt^3/3}$ can be brought inside the second partial derivative with respect to x because t is considered to be a constant.

$$\frac{\partial}{\partial t}(e^{bt^3/3}u) - k\frac{\partial^2}{\partial x^2}(e^{bt^3/3}u) = 0$$

Bring the second term to the right.

$$\frac{\partial}{\partial t}(e^{bt^3/3}u) = k\frac{\partial^2}{\partial x^2}(e^{bt^3/3}u)$$

$e^{bt^3/3}u$ satisfies the diffusion equation, and its solution is thus

$$e^{bt^3/3}u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}}\phi(s)\,ds.$$

Therefore,

$$u(x,t) = \frac{e^{-bt^3/3}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}}\phi(s)\,ds.$$