Exercise 4

Solve the diffusion equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.

Solution

Solution by the Similarity Method

We have to solve the initial value problem,

$$u_t = ku_{xx}, \quad u(x,0) = \phi(x). \tag{1}$$

In order to do so, we'll solve for the Green's function $G(x,t)$ in the corresponding PDE,

$$G_t = kG_{xx}, \quad G(x,0) = \delta(x), \tag{2}$$

where $\delta(x)$, the Dirac delta function, is defined as

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}.$$

The reason we're solving the equation with the delta function is that it has the extremely useful "sifting" property,

$$\int_{-\infty}^{\infty} f(s)\delta(x-s)\,ds = f(x),$$

so the solution to the initial value problem (1) in terms of the Green's function is

$$u(x,t) = \int_{-\infty}^{\infty} G(x-s,t)\phi(s)\,ds.$$

This can be verified by substituting this form for u into (1). Now we will go about solving (2) for $G(x,t)$ by using the similarity method (also known as the combination of variables method). Because u is a dimensionless quantity (that is, it yields a pure number with no units) the variables x, t, and k have to appear in the solution in a dimensionless combination. x has units of meters, t has units of seconds, and k has units of meters^2/second, so the combination of variables has to be

$$\frac{x^2}{kt}$$

or any convenient multiple or power thereof. Therefore,

$$u = u(\eta), \quad \text{where we choose } \eta = \frac{x}{\sqrt{kt}}.$$

We choose this particular form for η so the process of getting the final answer is smoother. We're trying to solve (2) for G, though, and G is not dimensionless; as can be seen from the initial condition, it has the same dimensions as $\delta(x)$. $\delta(x)$ has the inverse dimension of its argument, so G has dimensions of meters^{-1}. Thus, G has to be of the form,

$$G(x,t) = \frac{1}{\sqrt{kt}} g\left(\frac{x}{\sqrt{kt}}\right),$$

where g is an arbitrary function. In order to determine g, we have to plug this form into (2) and solve the resulting ODE. To start, write the expressions for G_t and G_{xx}.

$$\begin{aligned}\frac{\partial G}{\partial t} &= -\frac{1}{2\sqrt{kt^3}}g + \frac{1}{\sqrt{kt}}\left(-\frac{x}{2\sqrt{kt^3}}\right)g' \\ \frac{\partial G}{\partial x} &= \frac{1}{\sqrt{kt}} \cdot \frac{1}{\sqrt{kt}}g' = \frac{1}{kt}g' \\ \frac{\partial^2 G}{\partial x^2} &= \frac{1}{kt} \cdot \frac{1}{\sqrt{kt}}g'' = \frac{1}{\sqrt{k^3t^3}}g''\end{aligned}$$

Substituting these expressions into (2) gives

$$-\frac{1}{2\sqrt{kt^3}}g - \frac{1}{2\sqrt{kt^3}} \cdot \frac{x}{\sqrt{kt}}g' = \frac{1}{\sqrt{kt^3}}g''.$$

Cancel common terms and move everything to one side.

$$g'' + \frac{1}{2}\frac{x}{\sqrt{kt}}g' + \frac{1}{2}g = 0$$

Use the combination variable η.

$$g'' + \frac{\eta}{2}g' + \frac{1}{2}g = 0$$

The last two terms on the left side can be written as one using the product rule.

$$g'' + \left(\frac{\eta}{2}g\right)' = 0$$

Integrate both sides of the equation.

$$g' + \frac{\eta}{2}g = C_1$$

This is an inhomogeneous first-order linear differential equation that can be solved with an integrating factor. The integrating factor is

$$I = e^{\int \frac{\eta}{2}\,d\eta} = e^{\frac{\eta^2}{4}}.$$

Multiply both sides by I.

$$e^{\frac{\eta^2}{4}}g' + \frac{\eta}{2}e^{\frac{\eta^2}{4}}g = C_1e^{\frac{\eta^2}{4}}$$

The two terms on the left side can be written as one using the product rule.

$$\left(e^{\frac{\eta^2}{4}}g\right)' = C_1e^{\frac{\eta^2}{4}}$$

Integrate both sides of the equation a second time.

$$e^{\frac{\eta^2}{4}}g = \int^{\eta} C_1e^{\frac{s^2}{4}}\,ds + C_2$$

Hence, the arbitrary function g is

$$g(\eta) = e^{-\frac{\eta^2}{4}}\left[C_1\int^{\eta} e^{\frac{s^2}{4}}\,ds + C_2\right],$$

and consequently, the Green's function is

$$G = \frac{1}{\sqrt{kt}} g(\eta) = \frac{e^{-\frac{\eta^2}{4}}}{\sqrt{kt}} \left[C_1 \int^{\eta} e^{\frac{s^2}{4}} \, ds + C_2 \right]. \tag{3}$$

The next order of business is to determine the constants of integration, C_1 and C_2. We need to return to the diffusion equation and the initial condition in (2) to figure these out.

$$G_t = kG_{xx}$$

Integrate both sides of the equation with respect to x over the whole line.

$$\int_{-\infty}^{\infty} G_t \, dx = \int_{-\infty}^{\infty} kG_{xx} \, dx$$

Take out the time derivative from the left side and evaluate the right side.

$$\frac{d}{dt} \int_{-\infty}^{\infty} G \, dx = kG_x|_{-\infty}^{\infty}$$

We assume that G and G_x tend to 0 as $x \to \pm\infty$, so we have

$$\frac{d}{dt} \int_{-\infty}^{\infty} G \, dx = 0.$$

This implies that the quantity,

$$\int_{-\infty}^{\infty} G \, dx,$$

remains constant for all time. Initially $G(x, 0) = \delta(x)$, so

$$\int_{-\infty}^{\infty} G(x,t) \, dx = \int_{-\infty}^{\infty} G(x,0) \, dx = \int_{-\infty}^{\infty} \delta(x) \, dx = 1. \tag{4}$$

In order for this integral to converge, C_1 has to be 0. In terms of x and t, (3) becomes

$$G(x,t) = \frac{C_2}{\sqrt{kt}} e^{-\frac{x^2}{4kt}}.$$

C_2 can be thought of as a normalization constant that we determine by plugging into (4).

$$\int_{-\infty}^{\infty} G(x,t) \, dx = \int_{-\infty}^{\infty} \frac{C_2}{\sqrt{kt}} e^{-\frac{x^2}{4kt}} \, dx = 1$$

Make the following substitution to solve the integral.

$$v = \frac{x}{\sqrt{4kt}}$$
$$dv = \frac{dx}{\sqrt{4kt}} \quad \to \quad 2\,dv = \frac{dx}{\sqrt{kt}}$$

The integral becomes

$$2C_2 \int_{-\infty}^{\infty} e^{-v^2} \, dv = 1,$$

and it evaluates to $\sqrt{\pi}$.

$$2C_2\sqrt{\pi} = 1$$

Solving for C_2 yields

$$C_2 = \frac{1}{\sqrt{4\pi}}.$$

Therefore, the Green's function is

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}},$$

and the solution to the initial value problem in (1) is

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}} \phi(s)\, ds. \tag{5}$$

$u(x,t)$ can be interpreted as the convolution of the initial condition with a Gaussian filter. At every point x, $u(x,t)$ is an averaged, or smoothed, version of the initial condition over an interval of width $\sqrt{kt}$. As t increases, the range of the filter grows and $u(x,t)$ becomes increasingly smooth over x. Any discontinuities or kinks that are present in the initial condition are smoothed out. In this exercise, the initial condition is

$$\phi(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}.$$

If we substitute this into the formula in (5), then we get

$$u(x,t) = \int_0^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt}} e^{-s}\, ds.$$

Combine the exponentials into one.

$$u(x,t) = \int_0^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-s)^2}{4kt} - s}\, ds.$$

The exponent E becomes the following.

$$\begin{aligned} E &= -\frac{(x-s)^2}{4kt} - s \\ &= \frac{-x^2 + 2xs - s^2 - 4kts}{4kt} \\ &= \frac{-x^2 - s^2 + (2x - 4kt)s - (x-2kt)^2 + (x-2kt)^2}{4kt} \\ &= \frac{-s^2 + 2(x-2kt)s - (x-2kt)^2 - x^2 + (x-2kt)^2}{4kt} \\ &= \frac{-[s-(x-2kt)]^2 - \cancel{x^2} + \cancel{x^2} - 4ktx + 4k^2t^2}{4kt} \\ &= \frac{4kt(kt-x) - (s-x+2kt)^2}{4kt} \\ &= kt - x - \frac{(s-x+2kt)^2}{4kt} \end{aligned}$$

So we have for $u(x,t)$:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_0^\infty e^{-\frac{(s-x+2kt)^2}{4kt}}\,ds.$$

Make the following substitution to solve the integral.

$$p = \frac{s-x+2kt}{\sqrt{4kt}} \quad \rightarrow \quad p^2 = \frac{(s-x+2kt)^2}{4kt}$$

$$dp = \frac{ds}{\sqrt{4kt}}$$

The integral becomes

$$u(x,t) = \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{\frac{-x+2kt}{\sqrt{4kt}}}^\infty e^{-p^2}\,dp.$$

Split up the integral into two to get desired limits of integration.

$$u(x,t) = \frac{1}{\sqrt{\pi}} e^{kt-x} \left(\int_{\frac{-x+2kt}{\sqrt{4kt}}}^0 e^{-p^2}\,dp + \int_0^\infty e^{-p^2}\,dp \right)$$

Switch the limits on the first integral and add a minus sign.

$$u(x,t) = \frac{1}{\sqrt{\pi}} e^{kt-x} \left(-\int_0^{\frac{-x+2kt}{\sqrt{4kt}}} e^{-p^2}\,dp + \int_0^\infty e^{-p^2}\,dp \right)$$

The error function, erf z, is defined as

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-p^2}\,dp,$$

so we can write the first integral in terms of it. Also, the second integral can be evaluated to $\sqrt{\pi}/2$.

$$u(x,t) = \frac{1}{\sqrt{\pi}} e^{kt-x} \left[-\frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{-x+2kt}{\sqrt{4kt}} \right) + \frac{\sqrt{\pi}}{2} \right]$$

Therefore,

$$u(x,t) = \frac{1}{2} e^{kt-x} \left[1 - \operatorname{erf}\left(\frac{-x+2kt}{\sqrt{4kt}} \right) \right].$$

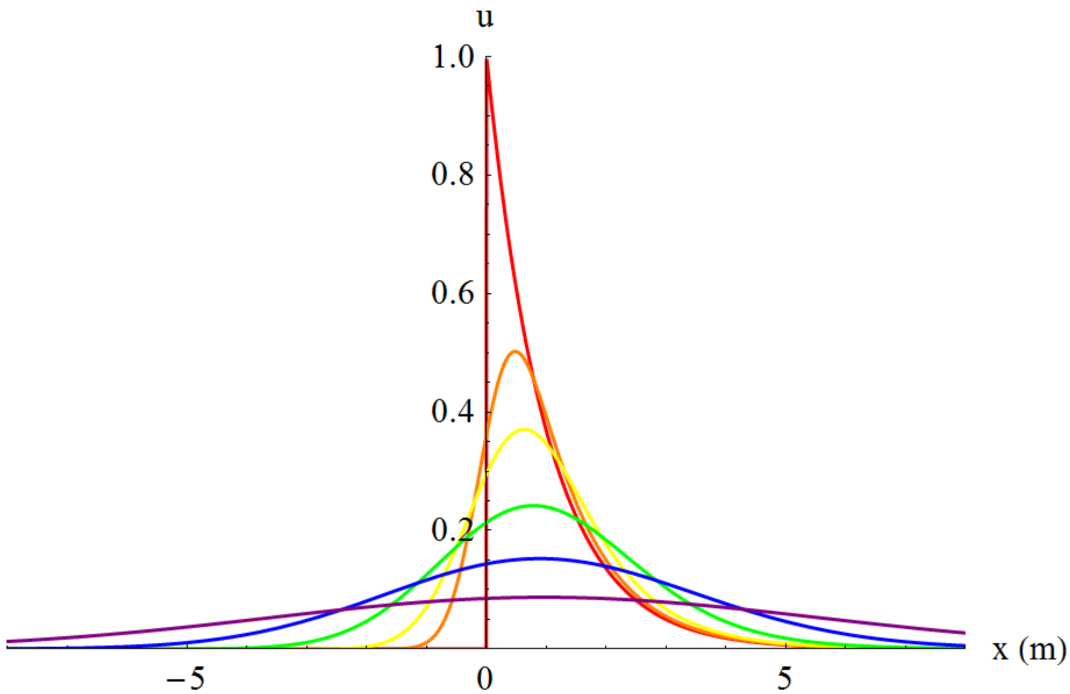

Figure 1: Plot of the solution $u(x,t)$ with $k = 1$ m^2/s for six different times: $t = 0$ s (red), $t = 0.1$ s (orange), $t = 0.3$ s (yellow), $t = 1$ s (green), $t = 3$ s (blue), and $t = 10$ s (purple).