Exercise 5

Prove properties (a) to (e) of the diffusion equation (1).

Solution

The aim here is to prove the invariance properties of the diffusion equation listed below.

(a) The translate $u(x-y,t)$ of any solution $u(x,t)$ is another solution for any fixed y.

(b) Any derivative of a solution is again a solution.

(c) A linear combination of solutions is again a solution.

(d) An integral of solutions is again a solution.

(e) If $u(x,t)$ is a solution, then so is the dilated function $u(\sqrt{a}x, at)$ for any $a>0$.

Proof of Property (a)

Show that $u(z=x-y,t)$ is a solution of the diffusion equation, $u_t = ku_{xx}$. Use the chain rule to write the derivatives of u.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial t} = u_z \cdot 0 + u_t \cdot 1 = u_t \\
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} = u_z \cdot 1 + u_t \cdot 0 = u_z \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial z}{\partial x}\frac{\partial}{\partial z}(u_z) = 1 \cdot u_{zz} = u_{zz}
\end{aligned}$$

Thus, if we substitute the expressions for u_t and u_{xx} into the diffusion equation, then we get

$$u_t = ku_{zz}.$$

This means that $u(x-y,t)$ satisfies the diffusion equation, and any translate of a solution is also a solution.

Proof of Property (b)

Show that any derivative of u (u_t, u_x, u_{xx}, etc.) satisfies the diffusion equation,

$$u_t = ku_{xx}.$$

Take any permutation of n derivatives with respect to x and t of both sides.

$$(u_t)_n = (ku_{xx})_n$$

Bring the constant out in front.

$$(u_t)_n = k(u_{xx})_n$$

Since it doesn't matter what order we take the derivatives, we can interchange them to our liking.

$$(u_n)_t = k(u_n)_{xx}$$

Therefore, u_n satisfies the diffusion equation, and any derivative of a solution is also a solution.

Proof of Property (c)

We have to show that any linear combination of solutions is also a solution to the diffusion equation, $u_t = ku_{xx}$. Suppose that $u_1(x,t)$, $u_2(x,t)$, $\ldots$, $u_n(x,t)$ are solutions to the diffusion equation. Then

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= k\frac{\partial^2 u_1}{\partial x^2}\\ \frac{\partial u_2}{\partial t} &= k\frac{\partial^2 u_2}{\partial x^2}\\ &\vdots\\ \frac{\partial u_n}{\partial t} &= k\frac{\partial^2 u_n}{\partial x^2}.\end{aligned}$$

Multiply both sides of the ith equation by an arbitrary constant a_i.

$$\begin{aligned}a_1\frac{\partial u_1}{\partial t} &= ka_1\frac{\partial^2 u_1}{\partial x^2}\\ a_2\frac{\partial u_2}{\partial t} &= ka_2\frac{\partial^2 u_2}{\partial x^2}\\ &\vdots\\ a_n\frac{\partial u_n}{\partial t} &= ka_n\frac{\partial^2 u_n}{\partial x^2}.\end{aligned}$$

Add all these equations together.

$$a_1\frac{\partial u_1}{\partial t} + a_2\frac{\partial u_2}{\partial t} + \cdots + a_n\frac{\partial u_n}{\partial t} = ka_1\frac{\partial^2 u_1}{\partial x^2} + ka_2\frac{\partial^2 u_2}{\partial x^2} + \cdots + ka_n\frac{\partial^2 u_n}{\partial x^2}$$

Factor k from the right side.

$$a_1\frac{\partial u_1}{\partial t} + a_2\frac{\partial u_2}{\partial t} + \cdots + a_n\frac{\partial u_n}{\partial t} = k\left(a_1\frac{\partial^2 u_1}{\partial x^2} + a_2\frac{\partial^2 u_2}{\partial x^2} + \cdots + a_n\frac{\partial^2 u_n}{\partial x^2}\right)$$

The constants can be put inside the derivatives.

$$\frac{\partial}{\partial t}(a_1u_1) + \frac{\partial}{\partial t}(a_2u_2) + \cdots + \frac{\partial}{\partial t}(a_nu_n) = k\left[\frac{\partial^2}{\partial x^2}(a_1u_1) + \frac{\partial^2}{\partial x^2}(a_2u_2) + \cdots + \frac{\partial^2}{\partial x^2}(a_nu_n)\right]$$

The sum of the derivatives is equal to the derivative of the sum, so we can factor out the derivatives from each side.

$$\frac{\partial}{\partial t}(a_1u_1 + a_2u_2 + \cdots + a_nu_n) = k\frac{\partial^2}{\partial x^2}(a_1u_1 + a_2u_2 + \cdots + a_nu_n)$$

Therefore, the linear combination of solutions satisfies the diffusion equation and is a solution as well.

Proof of Property (d)

Here we have to show that any integral of a solution to the diffusion equation is also a solution. Suppose then that $u(x,t)$ is a solution, that is,

$$u_t = ku_{xx}.$$

According to property (a), the translate $u(x-s,t)$ is a solution as well.

$$\frac{\partial u}{\partial t}(x-s,t) = k\frac{\partial^2 u}{\partial x^2}(x-s,t)$$

Multiply both sides of the equation by an integrable function $f(s)$.

$$\frac{\partial u}{\partial t}(x-s,t)f(s) = k\frac{\partial^2 u}{\partial x^2}(x-s,t)f(s)$$

Integrate both sides with respect to s over the whole line.

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x-s,t)f(s)\,ds = \int_{-\infty}^{\infty} k\frac{\partial^2 u}{\partial x^2}(x-s,t)f(s)\,ds$$

The constant k can be pulled in front of the integral.

$$\int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x-s,t)f(s)\,ds = k\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}(x-s,t)f(s)\,ds$$

Since the variable of integration is s, the partial derivatives with respect to t and x can be pulled in front of the integrals.

$$\frac{\partial}{\partial t}\left[\int_{-\infty}^{\infty} u(x-s,t)f(s)\,ds\right] = k\frac{\partial^2}{\partial x^2}\left[\int_{-\infty}^{\infty} u(x-s,t)f(s)\,ds\right]$$

Therefore, the integral of a solution to the diffusion equation also satisfies the equation and is a solution as well.

Proof of Property (e)

Show that $u(y=\sqrt{a}x, z=at)$ is a solution of the diffusion equation, $u_t = ku_{xx}$. Use the chain rule to write the derivatives of u in terms of the new variables.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = u_y \cdot 0 + u_z \cdot a = au_z \\
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = u_y \cdot \sqrt{a} + u_z \cdot 0 = \sqrt{a}u_y \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial y}{\partial x}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z}\right)(\sqrt{a}u_y) = \left(\sqrt{a}\cdot\frac{\partial}{\partial y} + 0\cdot\frac{\partial}{\partial z}\right)(\sqrt{a}u_y) = au_{yy}
\end{aligned}$$

Thus, if we substitute the expressions for u_t and u_{xx} into the diffusion equation, then we get

$$au_z = kau_{yy}.$$

After cancelling a we see that

$$u_z = ku_{yy},$$

which means that $u(y=\sqrt{a}x, z=at)$ satisfies the diffusion equation and any dilated solution is also a solution.