Exercise 8

Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta \leq |x| < \infty} S(x,t) \to 0 \qquad \text{as } t \to 0.$$

[This means that the tail of $S(x,t)$ is "uniformly small".]

Solution

$S(x,t)$ is defined in the textbook to be the Green's function for the diffusion equation.

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

The maximum occurs where x is smallest in magnitude because of the negative sign in the exponent. Since $\delta \leq |x| < \infty$, we set $x^2 = \delta^2$.

$$S(\delta,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{\delta^2}{4kt}}$$

Now we have to show that the limit of $S(\delta,t)$ as $t \to 0$ is 0.

$$\lim_{t\to 0} S(\delta,t) = \lim_{t\to 0} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{\delta^2}{4kt}}$$

Bring the constants in front of the limit.

$$\lim_{t\to 0} S(\delta,t) = \frac{1}{\sqrt{4\pi k}} \lim_{t\to 0} \frac{1}{\sqrt{t}} e^{-\frac{\delta^2}{4kt}}$$

Plugging in $t = 0$ yields the indeterminate form $0/0$. However, applying L'Hôpital's rule doesn't lead to any simplification, so we have to try something else. Proceed by bringing $\sqrt{t}$ to the exponent of the exponential function.

$$\lim_{t\to 0} S(\delta,t) = \frac{1}{\sqrt{4\pi k}} \lim_{t\to 0} e^{-\frac{\delta^2}{4kt} - \ln\sqrt{t}}$$

Factor out the minus sign and bring the limit into the exponent.

$$\lim_{t\to 0} S(\delta,t) = \frac{1}{\sqrt{4\pi k}} e^{-\lim\limits_{t\to 0}\left(\frac{\delta^2}{4kt} + \ln\sqrt{t}\right)}$$

Plugging in $t = 0$ now yields the indeterminate form $\infty - \infty$, so factor out $\delta^2/4kt$ in order to make this a product. Also, change $\sqrt{t}$ to t by bringing a factor of $1/2$ in front.

$$\lim_{t\to 0}\left(\frac{\delta^2}{4kt} + \ln\sqrt{t}\right) = \lim_{t\to 0} \frac{\delta^2}{4kt}\left(1 + \frac{2kt}{\delta^2}\ln t\right)$$

Since the limit of a product is the product of the limits, we can write this as

$$\lim_{t\to 0}\left(\frac{\delta^2}{4kt} + \ln\sqrt{t}\right) = \left(\lim_{t\to 0}\frac{\delta^2}{4kt}\right)\left[\lim_{t\to 0}\left(1 + \frac{2kt}{\delta^2}\ln t\right)\right].$$

The limit of a sum is the sum of the limits, so

$$\lim_{t\to 0}\left(\frac{\delta^2}{4kt}+\ln\sqrt{t}\right)=\left(\lim_{t\to 0}\frac{\delta^2}{4kt}\right)\left(\lim_{t\to 0}1+\lim_{t\to 0}\frac{2kt}{\delta^2}\ln t\right).$$

Bring the constants in front of the limits.

$$\lim_{t\to 0}\left(\frac{\delta^2}{4kt}+\ln\sqrt{t}\right)=\left(\frac{\delta^2}{4k}\lim_{t\to 0}\frac{1}{t}\right)\left(1+\frac{2k}{\delta^2}\lim_{t\to 0}t\ln t\right)$$

The last limit can be written as an indeterminate form ∞/∞.

$$\lim_{t\to 0}t\ln t=\lim_{t\to 0}\frac{\ln t}{\frac{1}{t}}$$

Apply L'Hôpital's rule and differentiate the numerator and denominator.

$$\lim_{t\to 0}t\ln t\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}}\lim_{t\to 0}\frac{\frac{1}{t}}{-\frac{1}{t^2}}=\lim_{t\to 0}(-t)=0$$

Thus,

$$\lim_{t\to 0}\left(\frac{\delta^2}{4kt}+\ln\sqrt{t}\right)=\frac{\delta^2}{4k}\lim_{t\to 0}\frac{1}{t}\to\infty.$$

This means that

$$\lim_{t\to 0}S(\delta,t)\to\frac{1}{\sqrt{4\pi k}}e^{-\infty}=0.$$

Therefore, for any fixed $\delta>0$,

$$\max_{\delta\le|x|<\infty}S(x,t)\to 0\qquad\text{as }t\to 0.$$