Exercise 1

Show that there is no maximum principle for the wave equation.

Solution[1]

Suppose that there is a maximum principle for the wave equation $u_{tt} = c^2 u_{xx}$. Let $u(x, t)$ be a solution to the wave equation in a rectangle $0 \leq x \leq L$ and $0 \leq t \leq T$. Then the maximum of $u(x, t)$ is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = L$). Consider the particular solution,

$$u(x, t) = \sin x \sin ct,$$

on $0 < x < \pi$ and $0 < t < \pi/c$. We can show that this satisfies the wave equation.

$$\begin{aligned} u_x &= \cos x \sin ct \\ u_{xx} &= -\sin x \sin ct \\ u_t &= c \sin x \cos ct \\ u_{tt} &= -c^2 \sin x \sin ct \end{aligned}$$

So $u_{tt} = c^2 u_{xx}$. According to the maximum principle, the maximum of $u(x, t)$ is zero because

$$u(0, t) = 0$$

$$u(\pi, t) = 0$$

$$u(x, 0) = 0.$$

But $u(x, t)$ has a higher value when $x = \pi/2$ and $t = \pi/(2c)$, for example.

$$u\left(\frac{\pi}{2}, \frac{\pi}{2c}\right) = 1$$

Therefore, there is no maximum principle for the wave equation.

[1]Special thanks to L. Baker for suggesting a better particular solution. Sorry it took so long to fix.