Exercise 2

Consider a traveling wave $u(x,t) = f(x - at)$ where f is a given function of one variable.

(a) If it is a solution of the wave equation, show that the speed must be $a \pm c$ (unless f is a linear function).

(b) If it is a solution of the diffusion equation, find f and show that the speed a is arbitrary.

Solution

Part (a)

If $u(x,t) = f(x - at)$ is a solution to the wave equation, then it has to satisfy $u_{tt} = c^2 u_{xx}$.

$$\begin{aligned} u_t &= (-a) \cdot f' \\ u_{tt} &= a^2 \cdot f'' \\ u_x &= f' \\ u_{xx} &= f'' \end{aligned}$$

Substituting these expressions for the terms in the PDE yields

$$a^2 f'' = c^2 f''.$$

This implies that

$$a^2 = c^2$$

or

$$a = \pm c.$$

If $f(x - at)$ is linear, then $f'' = 0$ and a^2 need not equal c^2.

Part (b)

If $u(x,t) = f(x - at)$ is a solution to the diffusion equation, then it has to satisfy $u_t = k u_{xx}$.

$$\begin{aligned} u_t &= (-a) \cdot f' \\ u_x &= f' \\ u_{xx} &= f'' \end{aligned}$$

Substituting these expressions for the terms in the PDE yields

$$-af' = kf''.$$

Rewrite this as

$$-\frac{a}{k} = \frac{f''}{f'} = \frac{d \ln f'}{d\xi},$$

where $\xi = x - at$. Integrate both sides once with respect to ξ.

$$-\frac{a}{k}\xi + C = \ln f'$$

Exponentiate both sides.

$$e^{-\frac{a}{k}\xi+C} = f'$$

Introduce a new constant of integration $C_1 = e^C$.

$$C_1 e^{-\frac{a}{k}\xi} = f'$$

Integrate both sides with respect to ξ a second time.

$$f(\xi) = C_3 e^{-\frac{a}{k}\xi} + C_2,$$

where C_2 and $C_3 = -kC_1/a$ are other arbitrary constants. Now change back to the original variables, x and t.

$$f(x-at) = C_3 e^{-\frac{a}{k}(x-at)} + C_2 \tag{1}$$

We can check that this satisfies the diffusion equation.

$$\begin{aligned} u_t &= C_3 e^{-\frac{a}{k}(x-at)} \cdot \left(\frac{a^2}{k}\right) \\ u_x &= C_3 e^{-\frac{a}{k}(x-at)} \cdot \left(-\frac{a}{k}\right) \\ u_{xx} &= C_3 e^{-\frac{a}{k}(x-at)} \cdot \left(\frac{a^2}{k^2}\right) \end{aligned}$$

Therefore, $u_t = ku_{xx}$, which means (1) is the correct solution for f. Note that there are no restrictions on a; that is, it is arbitrary.