Exercise 4

Consider the following problem with a Robin boundary condition:

$$\begin{array}{rlll} \text{DE:} & u_t = ku_{xx} & \text{on the half-line } 0 < x < \infty \text{ (and } 0 < t < \infty) & \\ \text{IC:} & u(x,0) = x & \text{for } t = 0 \text{ and } 0 < x < \infty & (*) \\ \text{BC:} & u_x(0,t) - 2u(0,t) = 0 & \text{for } x = 0 & \end{array}$$

The purpose of this exercise is to verify the solution formula for (*). Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

(a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?

(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy for $-\infty < x < \infty$?

(c) Show that $f'(x) - 2f(x)$ is an odd function (for $x \neq 0$).

(d) Use Exercise 2.4.11 to show that w is an odd function of x.

(e) Deduce that $v(x,t)$ satisfies (*) for $x > 0$. Assuming uniqueness, deduce that the solution of (*) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

Solution

Part (a)

We recognize $v(x,t)$ as the convolution of the Green's function for the diffusion equation and the initial condition.

$$v(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy$$

Therefore, $v(x,t)$ is the solution to the diffusion equation over the whole line with the given initial condition.

$$v_t = kv_{xx}, \quad v(x,0) = f(x) = \begin{cases} x & x > 0 \\ x + 1 - e^{2x} & x < 0 \end{cases}, \quad -\infty < x < \infty$$

Part (b)

Because any derivative of a solution to the diffusion equation is also a solution and any linear combination of solutions to the diffusion equation is also a solution, $w(x,t) = v_x(x,t) - 2v(x,t)$ is a solution to the diffusion equation. We can show this in a direct way as follows.

$$v_t = kv_{xx}$$

Differentiate both sides with respect to x.

$$(v_t)_x = k(v_{xx})_x \tag{1}$$

Multiply both sides of the original equation by 2.

$$2v_t = 2kv_{xx} \tag{2}$$

Now subtract (2) from (1).

$$(v_t)_x - 2v_t = k(v_{xx})_x - 2kv_{xx}$$

Change the order of differentiation in the first term on the left and the first term on the right.

$$(v_x)_t - 2v_t = k(v_x)_{xx} - 2kv_{xx}$$

Factor the operator from both sides.

$$(v_x - 2v)_t = k(v_x - 2v)_{xx}$$

Therefore, $w = v_x - 2v$ satisfies the diffusion equation $w_t = kw_{xx}$. The initial condition for it is $w(x,0) = v_x(x,0) - 2v(x,0)$.

$$v(x,0) = f(x) = \begin{cases} x & x > 0 \\ x + 1 - e^{2x} & x < 0 \end{cases} \quad \rightarrow \quad v_x(x,0) = f'(x) = \begin{cases} 1 & x > 0 \\ 1 - 2e^{2x} & x < 0 \end{cases}$$

Thus,

$$w(x,0) = f'(x) - 2f(x) = \begin{cases} 1 - 2x & x > 0 \\ -1 - 2x & x < 0 \end{cases}.$$

Part (c)

$w(x,0) = f'(x) - 2f(x)$ is an odd function if $w(-x,0) = -w(x,0)$. In part (b) we calculated $w(x,0)$ and wrote it as a piecewise function. Plug in $-x$ for x now.

$$w(-x,0) = \begin{cases} 1 - 2(-x) & -x > 0 \\ -1 - 2(-x) & -x < 0 \end{cases} = \begin{cases} 1 + 2x & x < 0 \\ -1 + 2x & x > 0 \end{cases} = \begin{cases} -(1 - 2x) & x > 0 \\ -(-1 - 2x) & x < 0 \end{cases} = -w(x,0)$$

Therefore, $w(x,0) = f'(x) - 2f(x)$ is an odd function.

Part (d)

According to Exercise 2.4.11, if the initial condition is an odd function of x, then the solution to the diffusion equation is also an odd function of x. In part (c) $w(x,0)$ was proven to be odd, so by Exercise 2.4.11, $w(x,t)$ has to be odd in x as well: $w(-x,t) = -w(x,t)$.

Part (e)

Since $w(x,t)$ is odd, the boundary condition $w(0,t) = 0$ will be satsified automatically, and the corresponding problem on the half-line can be solved by taking the restriction $x > 0$. According to section 2.4, the solution to

$$w_t = kw_{xx}, \quad w(x,0) = f'(x) - 2f(x), \quad -\infty < x < \infty$$

is given by

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} [f'(s) - 2f(s)]\, ds.$$

Now that we know w, we can solve for v by using the original substitution $w = v_x - 2v$.

$$v_x - 2v = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} [f'(s) - 2f(s)]\, ds$$

If we distribute the integral to both terms, the solution will become apparent.

$$v_x - 2v = \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f'(s)\, ds}_{= v_x(x,t)} - 2 \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\, ds}_{= v(x,t)}$$

The first term on the right is the solution to $(v_x)_t = k(v_x)_{xx}$ on the whole line with the initial condition $v_x(x,0) = f'(x)$, and the second term on the right (excluding the -2) is the solution to $v_t = kv_{xx}$ on the whole line with the initial condition $v(x,0) = f(x)$. That is,

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\, ds.$$

The restriction of $v(x,t)$ to $x > 0$ gives us the solution to the initial boundary value problem satisfied by $u(x,t)$. Because the solution to the problem is unique, this has to be the one and only solution for $u(x,t)$. Therefore,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\, ds, \quad x > 0.$$

Although this is not part of the question, if we plug in $f(x)$ and evaluate the integral, we can actually write $u(x,t)$ in terms of the complementary error function, which is a special function defined as

$$\operatorname{erfc} z = 1 - \operatorname{erf} z = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2}\, ds = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-s^2}\, ds,$$

as

$$u(x,t) = x + \frac{1}{2}\left[\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) - e^{2(x+2kt)} \operatorname{erfc}\left(\frac{x+4kt}{\sqrt{4kt}}\right)\right].$$

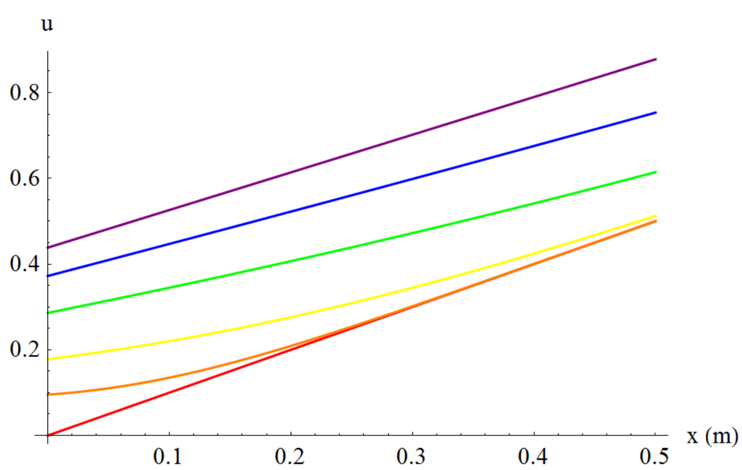

Figure 1: Plot of the solution $u(x,t)$ with $k = 1$ m^2/s for six different times: $t = 0$ s (red), $t = 0.01$ s (orange), $t = 0.05$ s (yellow), $t = 0.25$ s (green), $t = 1$ s (blue), and $t = 5$ s (purple).