Exercise 5

(a) Use the method of Exercise 4 to solve the Robin problem:

DE:	$u_t = ku_{xx}$	on the half-line $0 < x < \infty$ (and $0 < t < \infty$)
IC:	$u(x,0) = x$	for $t = 0$ and $0 < x < \infty$
BC:	$u_x(0,t) - hu(0,t) = 0$	for $x = 0$

where h is a constant.

(b) Generalize the method to the case of general initial data $\phi(x)$.

Solution

Part (a)

Consider the same PDE over the whole line ($-\infty < x < \infty$) with the initial condition $f(x)$ to be determined.

$$v_t = kv_{xx}, \quad v(x,0) = f(x), \quad -\infty < x < \infty$$

We now make the substitution $w = v_x - hv$. The point of doing this is to make it so that the boundary condition has only a single term on the left, $w(0,t)$. We want to choose $f(x)$ so that $w(0,t)$ is an odd function. It has to match the initial condition for u when $x > 0$, but we are free to choose whatever we want for $f(x)$ when $x < 0$. That is,

$$v(x,0) = f(x) = \begin{cases} x & x > 0 \\ g(x) & x < 0 \end{cases}, \quad -\infty < x < \infty.$$

$g(x)$ will be determined in a moment, but right now we want to see what PDE and initial condition $w(x,t)$ satisfies. Because any derivative of a solution to the diffusion equation is also a solution and any linear combination of solutions to the diffusion equation is also a solution, $w(x,t) = v_x(x,t) - hv(x,t)$ is a solution to the diffusion equation. We can show this in a direct way as follows.

$$v_t = kv_{xx}$$

Differentiate both sides with respect to x.

$$(v_t)_x = k(v_{xx})_x \tag{1}$$

Multiply both sides of the original equation by h.

$$hv_t = hkv_{xx} \tag{2}$$

Now subtract (2) from (1).

$$(v_t)_x - hv_t = k(v_{xx})_x - hkv_{xx}$$

Change the order of differentiation in the first term on the left and the first term on the right.

$$(v_x)_t - hv_t = k(v_x)_{xx} - hkv_{xx}$$

Factor the operator from both sides.

$$(v_x - hv)_t = k(v_x - hv)_{xx}$$

Therefore, $w = v_x - hv$ satisfies the diffusion equation $w_t = kw_{xx}$. The initial condition for it is $w(x, 0) = v_x(x, 0) - hv(x, 0)$. With this condition, $g(x)$ can be determined.

$$v(x,0) = f(x) = \begin{cases} x & x > 0 \\ g(x) & x < 0 \end{cases} \quad \rightarrow \quad v_x(x,0) = f'(x) = \begin{cases} 1 & x > 0 \\ g'(x) & x < 0 \end{cases}$$

Thus,

$$w(x,0) = f'(x) - hf(x) = \begin{cases} 1 - hx & x > 0 \\ g'(x) - hg(x) & x < 0 \end{cases}.$$

We want to choose $g(x)$ so that it makes $w(x, 0)$ odd, so the equation we have to solve for it is

$$g'(x) - hg(x) = -1 - hx.$$

This is a first-order inhomogeneous ODE that can be solved by multiplying both sides by an integrating factor.

$$I = e^{\int^x -h\,ds} = e^{-hx}$$

Proceed with the multiplication.

$$e^{-hx}g'(x) - he^{-hx}g(x) = -(1 + hx)e^{-hx}.$$

Recognize that the left side can be written as the derivative of the product $e^{-hx}g$.

$$\frac{d}{dx}(e^{-hx}g) = -(1 + hx)e^{-hx}$$

Integrate both sides now.

$$e^{-hx}g = \left(x + \frac{2}{h}\right)e^{-hx} + C$$

The condition to determine C is that $g(x)$ has to be 0 when $x = 0$ since we need $w(x, 0)$ to be odd. This means

$$0 = \frac{2}{h} + C \quad \rightarrow \quad C = -\frac{2}{h}.$$

Thus,

$$g(x) = x + \frac{2}{h} - \frac{2}{h}e^{hx}.$$

Now we'll show that $w(x, 0)$ is odd. Plug in $-x$ for x now.

$$w(-x,0) = \begin{cases} 1 - h(-x) & -x > 0 \\ -1 - h(-x) & -x < 0 \end{cases} = \begin{cases} 1 + hx & x < 0 \\ -1 + hx & x > 0 \end{cases} = \begin{cases} -(1 - hx) & x > 0 \\ -(-1 - hx) & x < 0 \end{cases} = -w(x,0)$$

Therefore, $w(x, 0) = f'(x) - hf(x)$ is an odd function. According to Exercise 2.4.11, if the initial condition is an odd function of x, then the solution to the diffusion equation is also an odd function of x. This means that $w(x, t)$ has to be odd in x as well: $w(-x, t) = -w(x, t)$. Since $w(x, t)$ is odd, the boundary condition $w(0, t) = 0$ will be satsified automatically, and the corresponding problem on the half-line can be solved by taking the restriction $x > 0$. According to section 2.4, the solution to

$$w_t = kw_{xx}, \quad w(x,0) = f'(x) - hf(x), \quad -\infty < x < \infty$$

is given by

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} [f'(s) - hf(s)]\, ds.$$

Now that we know w, we can solve for v by using the original substitution $w = v_x - hv$.

$$v_x - hv = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} [f'(s) - hf(s)]\, ds$$

If we distribute the integral to both terms, the solution will become apparent.

$$v_x - hv = \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f'(s)\, ds}_{= v_x(x,t)} - h \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\, ds}_{= v(x,t)}$$

The first term on the right is the solution to $(v_x)_t = k(v_x)_{xx}$ on the whole line with the initial condition $v_x(x,0) = f'(x)$, and the second term on the right (excluding the $-h$) is the solution to $v_t = kv_{xx}$ on the whole line with the initial condition $v(x,0) = f(x)$. That is,

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\, ds.$$

The restriction of $v(x,t)$ to $x > 0$ gives us the solution to the initial boundary value problem satisfied by $u(x,t)$. Because the solution to the problem is unique, this has to be the one and only solution for $u(x,t)$. Therefore,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\, ds, \quad x > 0,$$

where

$$f(x) = \begin{cases} x & x > 0 \\ x + \frac{2}{h} - \frac{2}{h}e^{hx} & x < 0 \end{cases}.$$

If we plug in $f(x)$ and evaluate the integral, we can actually write $u(x,t)$ in closed form using the complementary error function, which is a special function defined as

$$\operatorname{erfc} z = 1 - \operatorname{erf} z = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2}\, ds = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-s^2}\, ds,$$

as

$$u(x,t) = x + \frac{1}{h}\left[\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) - e^{h(x+hkt)} \operatorname{erfc}\left(\frac{x+2hkt}{\sqrt{4kt}}\right)\right].$$

Note that if we set $h = 2$, we get the answer to the previous exercise. Also, $\operatorname{erfc} 0 = 1$ and $\operatorname{erfc} \infty = 0$. The rest of this part is devoted to deriving this closed-form solution (completely optional). Substitute $f(x)$ into the integral.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \left[\int_{-\infty}^{0} e^{-\frac{(x-s)^2}{4kt}} \left(s + \frac{2}{h} - \frac{2}{h}e^{hs}\right) ds + \int_0^{\infty} e^{-\frac{(x-s)^2}{4kt}} s\, ds\right]$$

Substitute $p = -s$ and $dp = -ds$ into the first integral and $p = s$ and $dp = ds$ into the second integral.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \left[-\int_{\infty}^{0} e^{-\frac{(x+p)^2}{4kt}} \left(-p + \frac{2}{h} - \frac{2}{h}e^{-hp}\right) dp + \int_0^{\infty} e^{-\frac{(x-p)^2}{4kt}} p\, dp\right]$$

Use the minus sign to switch the limits of integration in the first integral and then distribute the first integral.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\left[-\int_0^\infty pe^{-\frac{(x+p)^2}{4kt}}\,dp + \frac{2}{h}\int_0^\infty e^{-\frac{(x+p)^2}{4kt}}\,dp - \frac{2}{h}\int_0^\infty e^{-hp}e^{-\frac{(x+p)^2}{4kt}}\,dp + \int_0^\infty pe^{-\frac{(x-p)^2}{4kt}}\,dp\right]$$

Combine the exponentials in the third integral. Also, make the r-substitution below in the first and second integrals and make the q-substitution below in the fourth integral.

$$r = \frac{x+p}{\sqrt{4kt}} \quad \rightarrow \quad p = r\sqrt{4kt} - x \qquad\qquad q = \frac{x-p}{\sqrt{4kt}} \quad \rightarrow \quad p = x - q\sqrt{4kt}$$

$$dr = \frac{dp}{\sqrt{4kt}} \qquad\qquad dq = -\frac{dp}{\sqrt{4kt}}$$

The solution becomes

$$u(x,t) = \frac{1}{\sqrt{\pi}}\left[-\int_{\frac{x}{\sqrt{4kt}}}^\infty (r\sqrt{4kt} - x)e^{-r^2}\,dr + \frac{2}{h}\int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-r^2}\,dr - \int_{\frac{x}{\sqrt{4kt}}}^{-\infty}(x - q\sqrt{4kt})e^{-q^2}\,dq\right] - \frac{1}{\sqrt{4\pi kt}}\frac{2}{h}\int_0^\infty e^{-\frac{(x+p)^2}{4kt} - hp}\,dp.$$

The second integral can be written in terms of erfc. Distribute the first and third integrals now.

$$u(x,t) = \frac{1}{h}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) + \frac{1}{\sqrt{\pi}}\left(-\sqrt{4kt}\int_{\frac{x}{\sqrt{4kt}}}^\infty re^{-r^2}\,dr + x\int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-r^2}\,dr - x\int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-q^2}\,dq + \sqrt{4kt}\int_{\frac{x}{\sqrt{4kt}}}^{-\infty} qe^{-q^2}\,dq\right) - \frac{1}{\sqrt{4\pi kt}}\frac{2}{h}\int_0^\infty e^{-\frac{(x+p)^2}{4kt} - hp}\,dp$$

Evaluating the first and fourth integrals, we find they are equivalent, so they can be cancelled. Also, the second integral can be written in terms of erfc.

$$u(x,t) = \frac{1}{h}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) + \frac{x}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) + \frac{1}{\sqrt{\pi}}\left(\cancel{-\sqrt{4kt}\cdot\frac{1}{2}e^{-\frac{x^2}{4kt}}} - x\int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-q^2}\,dq + \cancel{\sqrt{4kt}\cdot\frac{1}{2}e^{-\frac{x^2}{4kt}}}\right) - \frac{1}{\sqrt{4\pi kt}}\frac{2}{h}\int_0^\infty e^{-\frac{(x+p)^2}{4kt} - hp}\,dp$$

Rewrite the remaining integral in parentheses.

$$u(x,t) = \frac{1}{h}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) + \frac{x}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) - \frac{x}{\sqrt{\pi}}\left(\int_\infty^{-\infty} e^{-q^2}\,dq - \int_\infty^{\frac{x}{\sqrt{4kt}}} e^{-q^2}\,dq\right) - \frac{1}{\sqrt{4\pi kt}}\frac{2}{h}\int_0^\infty e^{-\frac{(x+p)^2}{4kt} - hp}\,dp$$

Evaluate the first integral and use the minus sign to switch the limits of integration in the second one.

$$u(x,t) = \frac{1}{h}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) + \frac{x}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) - \frac{x}{\sqrt{\pi}}\left(-\sqrt{\pi} + \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-q^2}\,dq\right) - \frac{1}{\sqrt{4\pi kt}}\frac{2}{h}\int_0^\infty e^{-\frac{(x+p)^2}{4kt} - hp}\,dp$$

Write the integral in parentheses in terms of erfc.

$$u(x,t) = \frac{1}{h}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) + \cancel{\frac{x}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right)} + x - \cancel{\frac{x}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right)} - \frac{1}{\sqrt{4\pi kt}}\frac{2}{h}\int_0^\infty e^{-\frac{(x+p)^2}{4kt} - hp}\,dp$$

To evaluate the last integral, complete the square in the exponent.

$$\begin{aligned}
-\frac{(x+p)^2}{4kt} - hp &= \frac{-x^2 - 2xp - p^2}{4kt} - hp \\
&= \frac{-x^2 - 2xp - p^2 - 4hpkt}{4kt} \\
&= \frac{-x^2 - 2p(x+2hkt) - p^2}{4kt} \\
&= \frac{-x^2 + (x+2hkt)^2 - (x+2hkt)^2 - 2p(x+2hkt) - p^2}{4kt} \\
&= \frac{-x^2 + (x+2hkt)^2}{4kt} - \frac{(x+2hkt)^2 + 2p(x+2hkt) + p^2}{4kt} \\
&= \frac{-\cancel{x^2} + \cancel{x^2} + 4hxkt + 4h^2k^2t^2}{4kt} - \frac{(x+2hkt+p)^2}{4kt} \\
&= hx + h^2kt - \frac{(x+2hkt+p)^2}{4kt}
\end{aligned}$$

The solution becomes

$$u(x,t) = x + \frac{1}{h}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) - \frac{1}{\sqrt{4\pi kt}}\frac{2}{h}\int_0^\infty e^{h(x+hkt)}e^{-\frac{(p+x+2hkt)^2}{4kt}}\,dp.$$

Use the substitution below to simplify the last integral.

$$\begin{aligned}
\ell &= \frac{p + x + 2hkt}{\sqrt{4kt}} \\
d\ell &= \frac{dp}{\sqrt{4kt}}
\end{aligned}$$

It becomes

$$u(x,t) = x + \frac{1}{h}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) - \frac{1}{\sqrt{\pi}}\frac{2}{h}e^{h(x+hkt)}\int_{\frac{x+2hkt}{\sqrt{4kt}}}^\infty e^{-\ell^2}\,d\ell.$$

Rewrite the integral in terms of erfc, and this gives us the final answer.

$$u(x,t) = x + \frac{1}{h}\operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right) - \frac{1}{h}e^{h(x+hkt)}\operatorname{erfc}\left(\frac{x+2hkt}{\sqrt{4kt}}\right)$$

Part (b)

Consider the same PDE over the whole line ($-\infty < x < \infty$) with the initial condition $f(x)$ to be determined.

$$v_t = kv_{xx}, \quad v(x,0) = f(x), \quad -\infty < x < \infty$$

We now make the substitution $w = v_x - hv$. The point of doing this is to make it so that the boundary condition has only a single term on the left, $w(0,t)$. We want to choose $f(x)$ so that $w(0,t)$ is an odd function. It has to match the initial condition for u when $x > 0$, but we are free to choose whatever we want for $f(x)$ when $x < 0$. That is,

$$v(x,0) = f(x) = \begin{cases} \phi(x) & x > 0 \\ g(x) & x < 0 \end{cases}, \quad -\infty < x < \infty.$$

$g(x)$ will be determined in a moment, but right now we want to see what PDE and initial condition $w(x,t)$ satisfies. Because any derivative of a solution to the diffusion equation is also a solution and any linear combination of solutions to the diffusion equation is also a solution, $w(x,t) = v_x(x,t) - hv(x,t)$ is a solution to the diffusion equation. We can show this in a direct way as follows.

$$v_t = kv_{xx}$$

Differentiate both sides with respect to x.

$$(v_t)_x = k(v_{xx})_x \tag{3}$$

Multiply both sides of the original equation by h.

$$hv_t = hkv_{xx} \tag{4}$$

Now subtract (4) from (3).

$$(v_t)_x - hv_t = k(v_{xx})_x - hkv_{xx}$$

Change the order of differentiation in the first term on the left and the first term on the right.

$$(v_x)_t - hv_t = k(v_x)_{xx} - hkv_{xx}$$

Factor the operator from both sides.

$$(v_x - hv)_t = k(v_x - hv)_{xx}$$

Therefore, $w = v_x - hv$ satisfies the diffusion equation $w_t = kw_{xx}$. The initial condition for it is $w(x,0) = v_x(x,0) - hv(x,0)$. With this condition, $g(x)$ can be determined.

$$v(x,0) = f(x) = \begin{cases} \phi(x) & x > 0 \\ g(x) & x < 0 \end{cases} \quad \rightarrow \quad v_x(x,0) = f'(x) = \begin{cases} \phi'(x) & x > 0 \\ g'(x) & x < 0 \end{cases}$$

Thus,

$$w(x,0) = f'(x) - hf(x) = \begin{cases} \phi'(x) - h\phi(x) & x > 0 \\ g'(x) - hg(x) & x < 0 \end{cases}.$$

We want to choose $g(x)$ so that it makes $w(x,0)$ odd (odd extension), so the equation we have to solve for it is

$$g'(x) - hg(x) = -\phi'(-x) + h\phi(-x).$$

This is a first-order inhomogeneous ODE that can be solved by multiplying both sides by an integrating factor.

$$I = e^{\int^x -h\,ds} = e^{-hx}$$

Proceed with the multiplication.

$$e^{-hx}g'(x) - he^{-hx}g(x) = [-\phi'(-x) + h\phi(-x)]e^{-hx}.$$

Recognize that the left side can be written as the derivative of the product $e^{-hx}g$.

$$\frac{d}{dx}(e^{-hx}g) = -\phi'(-x)e^{-hx} + h\phi(-x)e^{-hx}$$

Integrate both sides now from 0 to x. The lower limit 0 is arbitrary so long as we add an appropriate constant of integration C. The condition to determine C is that $g(x)$ has to be 0 when $x = 0$. Also, remember that $x < 0$ in this ODE.

$$e^{-hx}g = -\int_0^x \phi'(-s)e^{-hs}\,ds + \int_0^x h\phi(-s)e^{-hs}\,ds + C$$

Let $p = -s$ and $dp = -ds$ in both integrals. Bring h in front of the second integral.

$$e^{-hx}g = \int_0^{-x} \phi'(p)e^{hp}\,dp - h\int_0^{-x} \phi(p)e^{hp}\,dp + C$$

Use integration by parts to move the derivative to the exponential in the first integral.

$$e^{-hx}g = \phi(p)e^{hp}\Big|_0^{-x} - \int_0^{-x} \phi(p)\frac{d}{dp}(e^{hp})\,dp - h\int_0^{-x} \phi(p)e^{hp}\,dp + C$$

Evaluate the first term at both limits and the derivative in the second term.

$$e^{-hx}g = \phi(-x)e^{-hx} - \phi(0) - h\int_0^{-x} \phi(p)e^{hp}\,dp - h\int_0^{-x} \phi(p)e^{hp}\,dp + C$$

Plugging in $g(0) = 0$ yields

$$0 = \phi(0) - \phi(0) + C,$$

which implies that $C = 0$. Solving for g gives us

$$g(x) = \phi(-x) - e^{hx}\left[2h\int_0^{-x} \phi(p)e^{hp}\,dp + \phi(0)\right]$$

$g(x)$ was chosen to make $w(x,0)$ odd, but we'll show that $w(x,0)$ is odd anyway. Plug in $-x$ for x now.

$$w(-x,0) = \begin{cases} \phi'(-x) - h\phi(-x) & -x > 0 \\ -\phi'(x) + h\phi(x) & -x < 0 \end{cases} = \begin{cases} -[-\phi'(-x) + h\phi(-x)] & x < 0 \\ -[\phi'(x) - h\phi(x)] & x > 0 \end{cases} = -w(x,0)$$

Therefore, $w(x,0) = f'(x) - hf(x)$ is an odd function. According to Exercise 2.4.11, if the initial condition is an odd function of x, then the solution to the diffusion equation is also an odd function of x. This means that $w(x,t)$ has to be odd in x as well: $w(-x,t) = -w(x,t)$. Since $w(x,t)$ is odd, the boundary condition $w(0,t) = 0$ will be satsified automatically, and the corresponding problem on the half-line can be solved by taking the restriction $x > 0$. According to section 2.4, the solution to

$$w_t = kw_{xx}, \quad w(x,0) = f'(x) - hf(x), \quad -\infty < x < \infty$$

is given by

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} [f'(s) - hf(s)]\, ds.$$

Now that we know w, we can solve for v by using the original substitution $w = v_x - hv$.

$$v_x - hv = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} [f'(s) - hf(s)]\, ds$$

If we distribute the integral to both terms, the solution will become apparent.

$$v_x - hv = \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f'(s)\, ds}_{= v_x(x,t)} - h \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\, ds}_{= v(x,t)}$$

The first term on the right is the solution to $(v_x)_t = k(v_x)_{xx}$ on the whole line with the initial condition $v_x(x,0) = f'(x)$, and the second term on the right (excluding the $-h$) is the solution to $v_t = kv_{xx}$ on the whole line with the initial condition $v(x,0) = f(x)$. That is,

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\, ds.$$

The restriction of $v(x,t)$ to $x > 0$ gives us the solution to the initial boundary value problem satisfied by $u(x,t)$. Because the solution to the problem is unique, this has to be the one and only solution for $u(x,t)$. Therefore,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\, ds, \quad x > 0,$$

where

$$f(x) = \begin{cases} \phi(x) & x > 0 \\ \phi(-x) - e^{hx}\left[2h \int_0^{-x} \phi(p) e^{hp}\, dp + \phi(0)\right] & x < 0 \end{cases}.$$

If we set $\phi(x) = x$ and $h = 2$, then we get the $f(x)$ of the previous exercise.

$$f(x) = \begin{cases} x & x > 0 \\ x + 1 - e^{2x} & x < 0 \end{cases}$$

This confirms the result.