Exercise 8

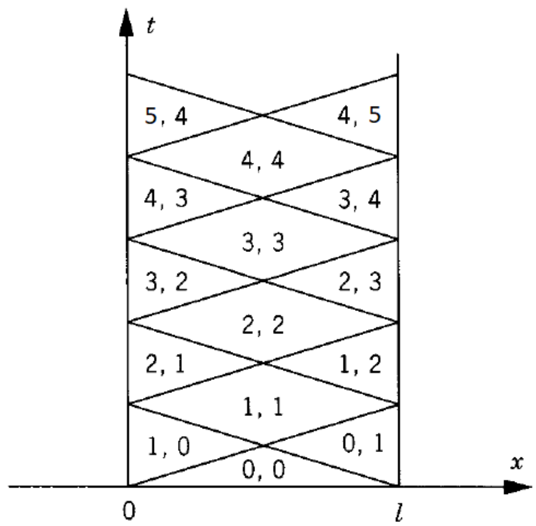
For the wave equation in a finite interval $(0, l)$ with Dirichlet conditions, explain the solution formula within each diamond-shaped region.

Solution

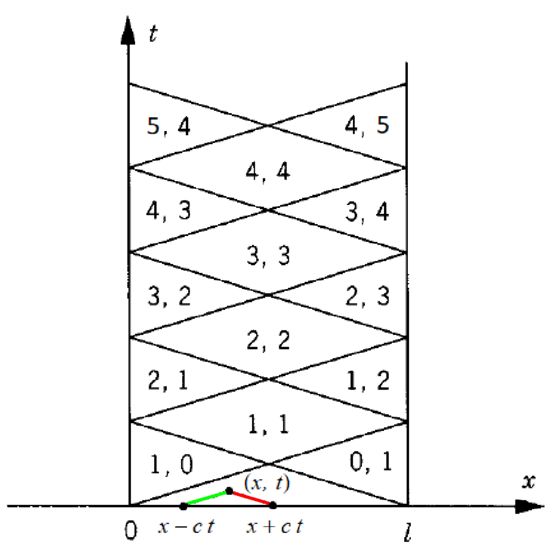
The goal here is to solve the initial boundary value problem,

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \qquad 0 < x < l,\ t > 0 \\ u(x,0) &= \phi(x) \quad u(0,t) = 0 \\ u_t(x,0) &= \psi(x) \quad u(l,t) = 0, \end{aligned}$$

in each of the diamonds illustrated below.



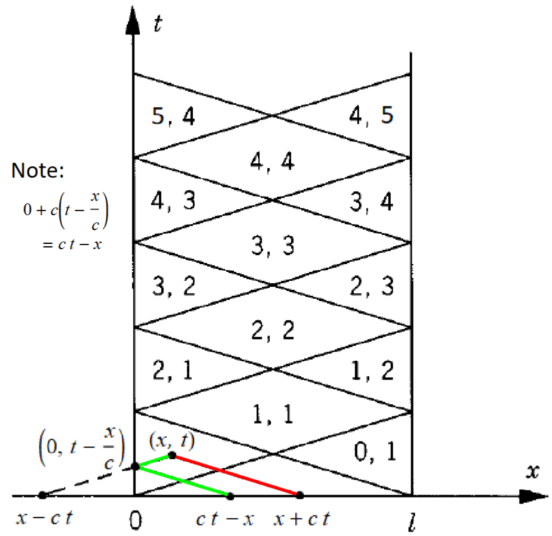
Diamond $\langle 0, 0 \rangle$



No reflections occur, so the solution in diamond $\langle 0, 0\rangle$ is given by d'Alembert's formula.

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(r)\, dr$$

Diamond $\langle 1, 0\rangle$



In this diamond a reflection occurs, so the method of reflection will be used. Consider the corresponding problem over the whole line, using the extensions for ϕ and ψ that are odd with respect to $x = 0$ and $x = l$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned} v_{tt} &= c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ v(x,0) &= \phi_{\text{ext}}(x) \\ v_t(x,0) &= \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{if } 0 < x < l \\ -\phi(-x) & \text{if } -l < x < 0 \\ \phi_{\text{ext}}(x+2l) \end{cases} \quad \text{and} \quad \psi_{\text{ext}}(x) = \begin{cases} \psi(x) & \text{if } 0 < x < l \\ -\psi(-x) & \text{if } -l < x < 0 \\ \psi_{\text{ext}}(x+2l) \end{cases}.$$

The solution for v is given by d'Alembert's formula.

$$v(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds$$

u is obtained by restricting this solution to $0 < x < l$.

$$u(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds, \quad 0 < x < l$$

Our task now is to write this formula in terms of the given functions, ϕ and ψ. There is one reflection on the left (so one minus sign), and there are no reflections on the right (so no minus signs).

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + (-1)\phi(ct-x)] + \frac{1}{2c}\left[\int_{x-ct}^{0} [-\psi(-s)]\,ds + \int_0^{x+ct} \psi(s)\,ds\right]$$

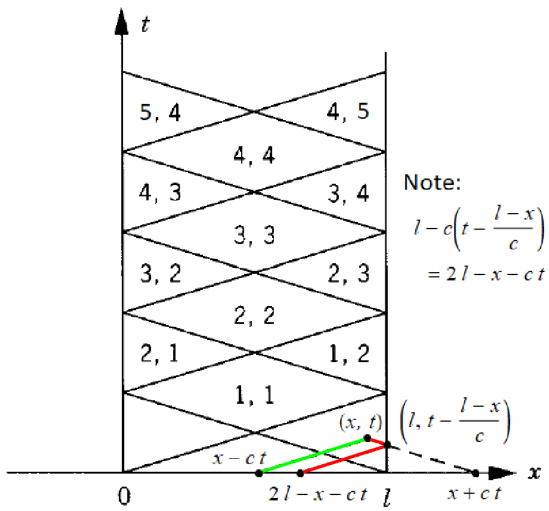
Substitute $r = -s$ in the first integral and $r = s$ in the second integral.

$$u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\left[\int_{ct-x}^{0} \psi(r)\,dr + \int_0^{x+ct} \psi(r)\,dr\right]$$

Therefore, the solution in diamond $\langle 1, 0\rangle$ is

$$u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(r)\,dr.$$

Diamond $\langle 0, 1\rangle$



In this diamond a reflection occurs, so the method of reflection will be used. Consider the corresponding problem over the whole line, using the extensions for ϕ and ψ that are odd with respect to $x = 0$ and $x = l$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned} &v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x,0) = \phi_{\text{ext}}(x) \\ &v_t(x,0) = \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

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The solution for v is given by d'Alembert's formula.

$$v(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds$$

u is obtained by restricting this solution to $0 < x < l$.

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Our task now is to write this formula in terms of the given functions, ϕ and ψ. There are no reflections on the left (so no minus signs), and there is one reflection on the right (so one minus sign).

$$u(x,t) = \frac{1}{2}[(-1)\phi(2l-x-ct) + \phi(x-ct)] + \frac{1}{2c}\left[\int_{x-ct}^{l} \psi(s)\,ds + \int_{l}^{x+ct} [-\psi(-s+2l)]\,ds\right]$$

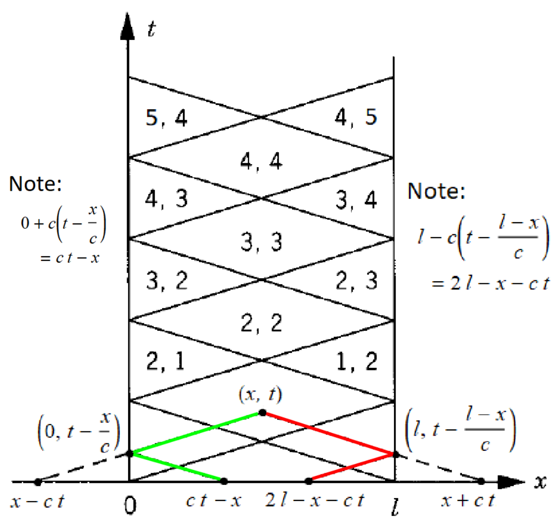
Substitute $r = s$ in the first integral and $r = -s + 2l$ in the second integral.

$$u(x,t) = \frac{1}{2}[-\phi(2l-x-ct) + \phi(x-ct)] + \frac{1}{2c}\left[\int_{x-ct}^{l} \psi(r)\,dr + \int_{l}^{2l-x-ct} \psi(r)\,dr\right]$$

Therefore, the solution in diamond $\langle 0,1\rangle$ is

$$u(x,t) = \frac{1}{2}[-\phi(2l-x-ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{2l-x-ct} \psi(r)\,dr.$$

Diamond $\langle 1,1\rangle$



In this diamond reflections occur, so the method of reflection will be used. Consider the corresponding problem over the whole line, using the extensions for ϕ and ψ that are odd with respect to $x = 0$ and $x = l$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned} &v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x,0) = \phi_{\text{ext}}(x) \\ &v_t(x,0) = \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

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The solution for v is given by d'Alembert's formula.

$$v(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds$$

u is obtained by restricting this solution to $0 < x < l$.

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Our task now is to write this formula in terms of the given functions, ϕ and ψ. There is one reflection on the left (so one minus sign), and there is one reflection on the right (so one minus sign).

$$u(x,t) = \frac{1}{2}[(-1)\phi(2l-x-ct) + (-1)\phi(ct-x)] \\ + \frac{1}{2c}\left[\int_{x-ct}^{0} [-\psi(-s)]\, ds + \int_{0}^{l} \psi(s)\, ds + \int_{l}^{x+ct} [-\psi(-s+2l)]\, ds\right]$$

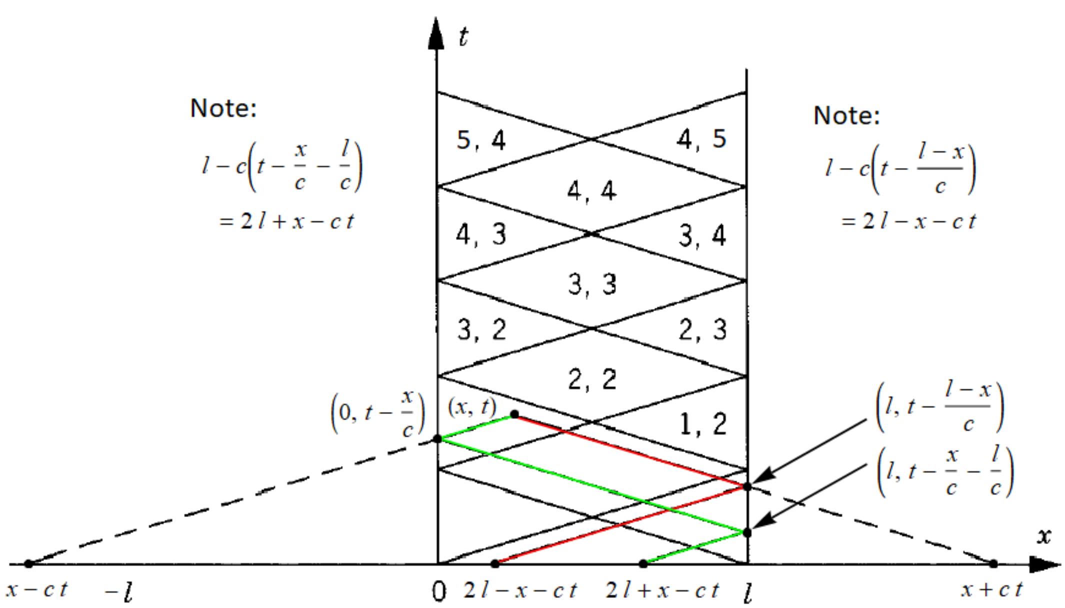
Substitute $r = -s$ in the first integral, $r = s$ in the second integral, and $r = -s + 2l$ in the third integral.

$$u(x,t) = \frac{1}{2}[-\phi(2l-x-ct) - \phi(ct-x)] + \frac{1}{2c}\left[\int_{ct-x}^{0} \psi(r)\, dr + \int_{0}^{l} \psi(r)\, dr + \int_{l}^{2l-x-ct} \psi(r)\, dr\right]$$

Therefore, the solution in diamond $\langle 1, 1\rangle$ is

$$u(x,t) = -\frac{1}{2}[\phi(2l-x-ct) + \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{2l-x-ct} \psi(r)\, dr.$$

Diamond $\langle 2, 1\rangle$



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$$\begin{aligned} &v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x,0) = \phi_{\text{ext}}(x) \\ &v_t(x,0) = \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

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The solution for v is given by d'Alembert's formula.

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u is obtained by restricting this solution to $0 < x < l$.

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Our task now is to write this formula in terms of the given functions, ϕ and ψ. There are two reflections on the way to $2l + x - ct$ (so two minus signs), and there is one reflection on the way to $2l - x - ct$ (so one minus sign).

$$\begin{aligned} u(x,t) = &\frac{1}{2}[(-1)\phi(2l - x - ct) + (-1)^2\phi(2l + x - ct)] \\ &+ \frac{1}{2c}\left[\int_{x-ct}^{-l} \psi(s+2l)\,ds + \int_{-l}^{0}[-\psi(-s)]\,ds + \int_0^l \psi(s)\,ds + \int_l^{x+ct}[-\psi(-s+2l)]\,ds\right] \end{aligned}$$

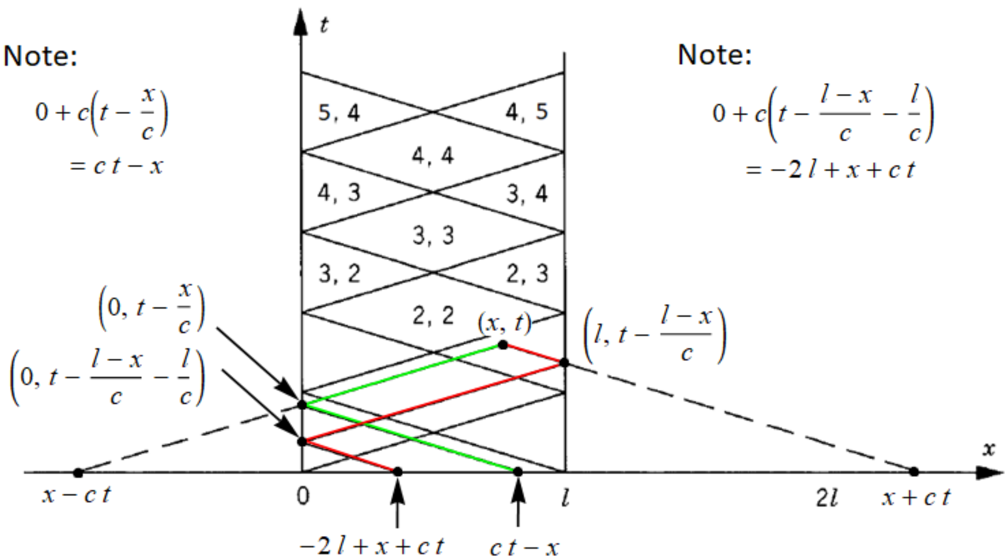
Substitute $r = s + 2l$ in the first integral, $r = -s$ in the second integral, $r = s$ in the third integral, and $r = -s + 2l$ in the fourth integral.

$$\begin{aligned} u(x,t) &= \frac{1}{2}[-\phi(2l - x - ct) + \phi(2l + x - ct)] \\ &\qquad + \frac{1}{2c}\left[\int_{2l+x-ct}^{l} \psi(r)\,dr + \cancel{\int_l^0 \psi(r)\,dr} + \cancel{\int_0^l \psi(r)\,dr} + \int_l^{2l-x-ct} \psi(r)\,dr\right] \\ &= \frac{1}{2}[-\phi(2l - x - ct) + \phi(2l + x - ct)] + \int_{2l+x-ct}^{2l-x-ct} \psi(r)\,dr \end{aligned}$$

Therefore, the solution in diamond $\langle 2, 1\rangle$ is

$$u(x,t) = \frac{1}{2}[-\phi(2l - x - ct) + \phi(2l + x - ct)] - \int_{2l-x-ct}^{2l+x-ct} \psi(r)\,dr.$$

Diamond $\langle 1, 2 \rangle$



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u is obtained by restricting this solution to $0 < x < l$.

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Our task now is to write this formula in terms of the given functions, ϕ and ψ. There is one reflection on the way to $ct - x$ (so one minus sign), and there are two reflections on the way to $-2l + x + ct$ (so two minus signs).

$$\begin{aligned} u(x, t) = {} & \frac{1}{2}[(-1)^2\phi(-2l + x + ct) + (-1)\phi(ct - x)] \\ & + \frac{1}{2c}\left[\int_{x-ct}^{0} [-\psi(-s)]\, ds + \int_0^l \psi(s)\, ds + \int_l^{2l} [-\psi(-s + 2l)]\, ds + \int_{2l}^{x+ct} \psi(s - 2l)\, ds\right] \end{aligned}$$

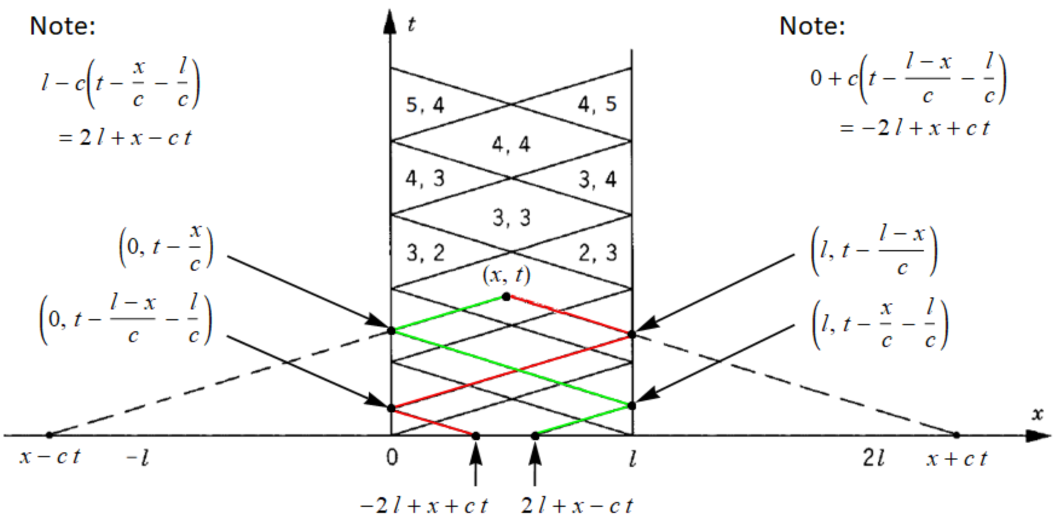
Substitute $r = -s$ in the first integral, $r = s$ in the second integral, $r = -s + 2l$ in the third integral, and $r = s - 2l$ in the fourth integral.

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\phi(-2l + x + ct) - \phi(ct - x)] \\ &\qquad + \frac{1}{2c}\left[\int_{ct-x}^{0} \psi(r)\,dr + \cancel{\int_0^l \psi(r)\,dr} + \cancel{\int_l^0 \psi(r)\,dr} + \int_0^{-2l+x+ct} \psi(r)\,dr\right] \\ &= \frac{1}{2}[\phi(-2l + x + ct) - \phi(ct - x)] + \int_{ct-x}^{-2l+x+ct} \psi(r)\,dr \end{aligned}$$

Therefore, the solution in diamond $\langle 1, 2\rangle$ is

$$u(x,t) = \frac{1}{2}[\phi(-2l + x + ct) - \phi(ct - x)] - \int_{-2l+x+ct}^{ct-x} \psi(r)\,dr.$$

Diamond $\langle 2, 2\rangle$



In this diamond reflections occur, so the method of reflection will be used. Consider the corresponding problem over the whole line, using the extensions for ϕ and ψ that are odd with respect to $x = 0$ and $x = l$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned} &v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x,0) = \phi_{\text{ext}}(x) \\ &v_t(x,0) = \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{if } 0 < x < l \\ -\phi(-x) & \text{if } -l < x < 0 \\ \phi_{\text{ext}}(x + 2l) \end{cases} \quad \text{and} \quad \psi_{\text{ext}}(x) = \begin{cases} \psi(x) & \text{if } 0 < x < l \\ -\psi(-x) & \text{if } -l < x < 0 \\ \psi_{\text{ext}}(x + 2l) \end{cases}.$$

The solution for v is given by d'Alembert's formula.

$$v(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds$$

u is obtained by restricting this solution to $0 < x < l$.

$$u(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds, \quad 0 < x < l$$

Our task now is to write this formula in terms of the given functions, ϕ and ψ. There are two reflections on the way to $2l + x - ct$ (so two minus signs), and there are two reflections on the way to $-2l + x + ct$ (so two minus signs).

$$\begin{aligned} u(x,t) = \frac{1}{2}&[(-1)^2\phi(-2l+x+ct) + (-1)^2\phi(2l+x-ct)] \\ &+ \frac{1}{2c}\left[\int_{x-ct}^{-l} \psi(s+2l)\,ds + \int_{-l}^{0}[-\psi(-s)]\,ds + \int_0^l \psi(s)\,ds \right. \\ &\qquad \left. + \int_l^{2l}[-\psi(-s+2l)]\,ds + \int_{2l}^{x+ct}\psi(s-2l)\,ds\right] \end{aligned}$$

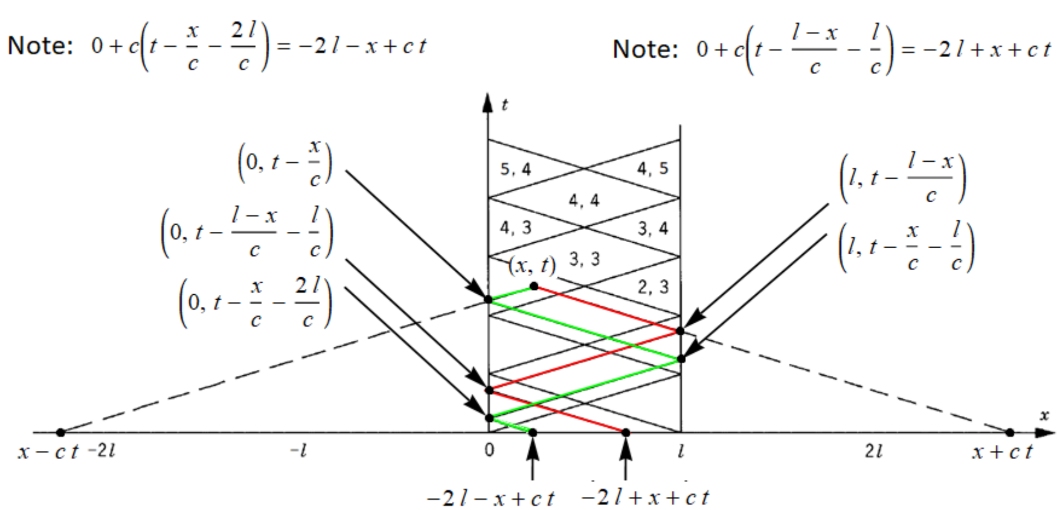
Substitute $r = s + 2l$ in the first integral, $r = -s$ in the second integral, $r = s$ in the third integral, $r = -s + 2l$ in the fourth integral, and $r = s - 2l$ in the fifth integral.

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\phi(-2l+x+ct) + \phi(2l+x-ct)] \\ &\qquad + \frac{1}{2c}\left[\int_{2l+x-ct}^{l} \psi(r)\,ds + \int_l^0 \psi(r)\,dr + \int_0^l \psi(r)\,dr + \int_l^0 \psi(r)\,dr + \int_0^{-2l+x+ct}\psi(r)\,dr\right] \\ &= \frac{1}{2}[\phi(-2l+x+ct) + \phi(2l+x-ct)] + \frac{1}{2c}\int_{2l+x-ct}^{-2l+x+ct}\psi(r)\,dr \end{aligned}$$

Therefore, the solution in diamond $\langle 2, 2\rangle$ is

$$u(x,t) = \frac{1}{2}[\phi(-2l+x+ct) + \phi(2l+x-ct)] - \frac{1}{2c}\int_{-2l+x+ct}^{2l+x-ct}\psi(r)\,dr.$$

Diamond $\langle 3, 2\rangle$



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$$\begin{aligned} & v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ & v(x,0) = \phi_{\text{ext}}(x) \\ & v_t(x,0) = \psi_{\text{ext}}(x) \end{aligned}$$

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The solution for v is given by d'Alembert's formula.

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u is obtained by restricting this solution to $0 < x < l$.

$$u(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds, \quad 0 < x < l$$

Our task now is to write this formula in terms of the given functions, ϕ and ψ. There are three reflections on the way to $-2l - x + ct$ (so three minus signs), and there are two reflections on the way to $-2l + x + ct$ (so two minus signs).

$$\begin{aligned} u(x,t) = \frac{1}{2}[(-1)^2\phi(-2l+x+ct) + (-1)^3\phi(-2l-x+ct)] \\ + \frac{1}{2c}\left[\int_{x-ct}^{-2l}[-\psi(-s-2l)]\,ds + \int_{-2l}^{-l}\psi(s+2l)\,ds + \int_{-l}^{0}[-\psi(-s)]\,ds \right. \\ \left. + \int_0^l \psi(s)\,ds + \int_l^{2l}[-\psi(-s+2l)]\,ds + \int_{2l}^{x+ct}\psi(s-2l)\,ds\right] \end{aligned}$$

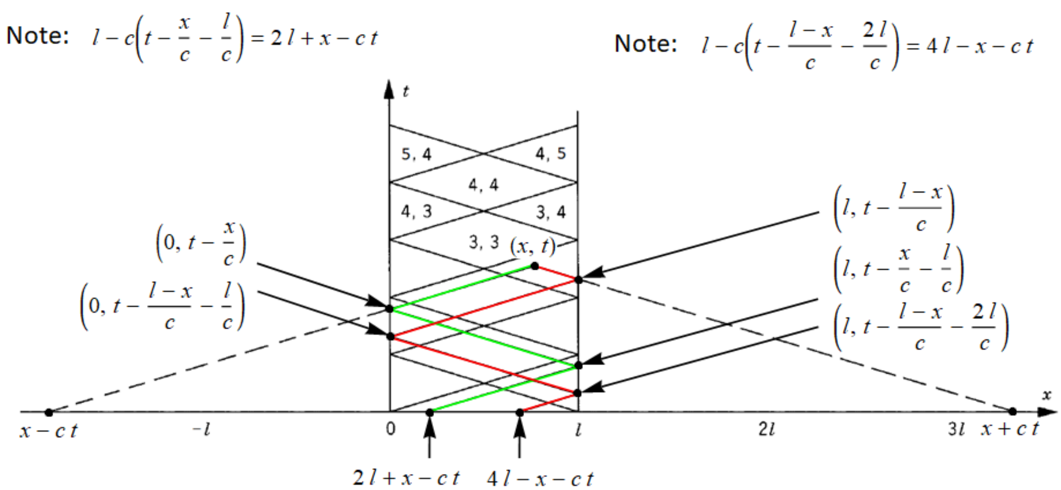
Substitute $r = -s - 2l$ in the first integral, $r = s + 2l$ in the second integral, $r = -s$ in the third integral, $r = s$ in the fourth integral, $r = -s + 2l$ in the fifth integral, and $r = s - 2l$ in the sixth integral.

$$\begin{aligned} u(x,t) = \frac{1}{2}[\phi(-2l+x+ct) - \phi(-2l-x+ct)] \\ + \frac{1}{2c}\left[\int_{-2l-x+ct}^{0}\psi(r)\,ds + \cancel{\int_0^l \psi(r)\,dr} + \cancel{\int_l^0 \psi(r)\,dr} \right. \\ \left. + \cancel{\int_0^l \psi(r)\,dr} + \cancel{\int_l^0 \psi(r)\,dr} + \int_0^{-2l+x+ct}\psi(r)\,dr\right] \end{aligned}$$

Therefore, the solution in diamond $\langle 3, 2\rangle$ is

$$u(x,t) = \frac{1}{2}[\phi(-2l+x+ct) - \phi(-2l-x+ct)] + \int_{-2l-x+ct}^{-2l+x+ct}\psi(r)\,dr.$$

Diamond $\langle 2, 3\rangle$



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$$\begin{aligned} &v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x, 0) = \phi_{\text{ext}}(x) \\ &v_t(x, 0) = \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{if } 0 < x < l \\ -\phi(-x) & \text{if } -l < x < 0 \\ \phi_{\text{ext}}(x + 2l) \end{cases} \quad \text{and} \quad \psi_{\text{ext}}(x) = \begin{cases} \psi(x) & \text{if } 0 < x < l \\ -\psi(-x) & \text{if } -l < x < 0 \\ \psi_{\text{ext}}(x + 2l) \end{cases}.$$

The solution for v is given by d'Alembert's formula.

$$v(x, t) = \frac{1}{2}[\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds$$

u is obtained by restricting this solution to $0 < x < l$.

$$u(x, t) = \frac{1}{2}[\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds, \quad 0 < x < l$$

Our task now is to write this formula in terms of the given functions, ϕ and ψ. There are two reflections on the way to $2l + x - ct$ (so two minus signs), and there are three reflections on the way to $4l - x - ct$ (so three minus signs).

$$\begin{aligned} u(x, t) = \frac{1}{2}&[(-1)^3\phi(4l - x - ct) + (-1)^2\phi(2l + x - ct)] \\ &+ \frac{1}{2c}\left[\int_{x-ct}^{-l} \psi(s + 2l)\, ds + \int_{-l}^{0}[-\psi(-s)]\, ds + \int_0^l \psi(s)\, ds \right. \\ &\left. + \int_l^{2l}[-\psi(-s + 2l)]\, ds + \int_{2l}^{3l} \psi(s - 2l)\, ds + \int_{3l}^{x+ct}[-\psi(-s + 4l)]\, ds\right] \end{aligned}$$

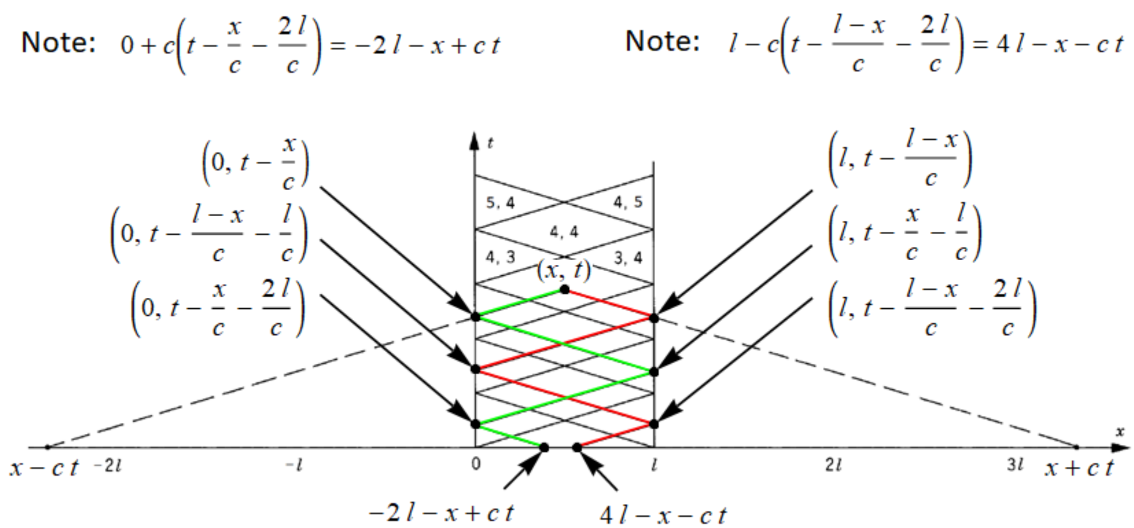
Substitute $r = s + 2l$ in the first integral, $r = -s$ in the second integral, $r = s$ in the third integral, $r = -s + 2l$ in the fourth integral, $r = s - 2l$ in the fifth integral, and $r = -s + 4l$ in the sixth integral.

$$\begin{aligned} u(x,t) = \frac{1}{2}[-\phi(4l - x - ct) + \phi(2l + x - ct)] \\ + \frac{1}{2c}\left[\int_{2l+x-ct}^{l} \psi(r)\,dr + \cancel{\int_{l}^{0} \psi(r)\,dr} + \cancel{\int_{0}^{l} \psi(r)\,dr} \right. \\ \left. + \cancel{\int_{l}^{0} \psi(r)\,dr} + \cancel{\int_{0}^{l} \psi(r)\,dr} + \int_{l}^{4l-x-ct} \psi(r)\,dr\right] \end{aligned}$$

Therefore, the solution in diamond $\langle 2, 3\rangle$ is

$$u(x,t) = \frac{1}{2}[-\phi(4l - x - ct) + \phi(2l + x - ct)] + \frac{1}{2c}\int_{2l+x-ct}^{4l-x-ct} \psi(r)\,dr.$$

Diamond $\langle 3, 3\rangle$



In this diamond reflections occur, so the method of reflection will be used. Consider the corresponding problem over the whole line, using the extensions for ϕ and ψ that are odd with respect to $x = 0$ and $x = l$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned} &v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x,0) = \phi_{\text{ext}}(x) \\ &v_t(x,0) = \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{if } 0 < x < l \\ -\phi(-x) & \text{if } -l < x < 0 \\ \phi_{\text{ext}}(x + 2l) & \end{cases} \quad \text{and} \quad \psi_{\text{ext}}(x) = \begin{cases} \psi(x) & \text{if } 0 < x < l \\ -\psi(-x) & \text{if } -l < x < 0 \\ \psi_{\text{ext}}(x + 2l) & \end{cases}.$$

The solution for v is given by d'Alembert's formula.

$$v(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds$$

u is obtained by restricting this solution to $0 < x < l$.

$$u(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds, \quad 0 < x < l$$

Our task now is to write this formula in terms of the given functions, ϕ and ψ. There are three reflections on the way to $-2l - x + ct$ (so three minus signs), and there are three reflections on the way to $4l - x - ct$ (so three minus signs).

$$\begin{aligned}u(x,t) = \frac{1}{2}&[(-1)^3\phi(4l-x-ct) + (-1)^3\phi(-2l-x+ct)] \\ &+ \frac{1}{2c}\left[\int_{x-ct}^{-2l}[-\psi(-s-2l)]\,ds + \int_{-2l}^{-l}\psi(s+2l)\,ds + \int_{-l}^{0}[-\psi(-s)]\,ds + \int_0^l \psi(s)\,ds\right. \\ &\qquad \left. + \int_l^{2l}[-\psi(-s+2l)]\,ds + \int_{2l}^{3l}\psi(s-2l)\,ds + \int_{3l}^{x+ct}[-\psi(-s+4l)]\,ds\right]\end{aligned}$$

Substitute $r = -s - 2l$ in the first integral, $r = s + 2l$ in the second integral, $r = -s$ in the third integral, $r = s$ in the fourth integral, $r = -s + 2l$ in the fifth integral, $r = s - 2l$ in the sixth integral, and $r = -s + 4l$ in the seventh integral.

$$\begin{aligned}u(x,t) = \frac{1}{2}&[-\phi(4l-x-ct) - \phi(-2l-x+ct)] \\ &+ \frac{1}{2c}\left[\int_{-2l-x+ct}^{0}\psi(r)\,dr + \cancel{\int_0^l \psi(r)\,dr} + \cancel{\int_l^0 \psi(r)\,dr} + \cancel{\int_0^l \psi(r)\,dr}\right. \\ &\qquad \left. + \cancel{\int_l^0 \psi(r)\,dr} + \int_0^l \psi(r)\,dr + \int_l^{4l-x-ct}\psi(r)\,dr\right]\end{aligned}$$

Therefore, the solution in diamond $\langle 3, 3\rangle$ is

$$u(x,t) = -\frac{1}{2}[\phi(4l-x-ct) + \phi(-2l-x+ct)] + \frac{1}{2c}\int_{-2l-x+ct}^{4l-x-ct}\psi(r)\,dr.$$

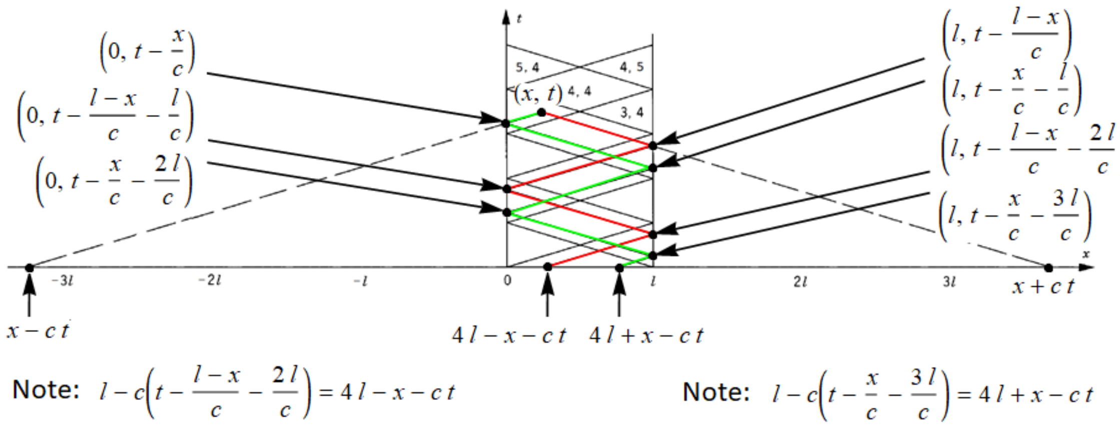
Diamond $\langle 4, 3\rangle$

In this diamond reflections occur, so the method of reflection will be used. Consider the corresponding problem over the whole line, using the extensions for ϕ and ψ that are odd with respect to $x = 0$ and $x = l$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned}&v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x,0) = \phi_{\text{ext}}(x) \\ &v_t(x,0) = \psi_{\text{ext}}(x)\end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

$$\phi_{\text{ext}}(x) = \begin{cases}\phi(x) & \text{if } 0 < x < l \\ -\phi(-x) & \text{if } -l < x < 0 \\ \phi_{\text{ext}}(x+2l) & \end{cases} \quad \text{and} \quad \psi_{\text{ext}}(x) = \begin{cases}\psi(x) & \text{if } 0 < x < l \\ -\psi(-x) & \text{if } -l < x < 0 \\ \psi_{\text{ext}}(x+2l) & \end{cases}.$$



The solution for v is given by d'Alembert's formula.

$$v(x,t)=\frac{1}{2}[\phi_{\text{ext}}(x+ct)+\phi_{\text{ext}}(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{\text{ext}}(s)\,ds$$

u is obtained by restricting this solution to $0<x<l$.

$$u(x,t)=\frac{1}{2}[\phi_{\text{ext}}(x+ct)+\phi_{\text{ext}}(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{\text{ext}}(s)\,ds,\quad 0<x<l$$

Our task now is to write this formula in terms of the given functions, ϕ and ψ. There are four reflections on the way to $4l+x-ct$ (so four minus signs), and there are three reflections on the way to $4l-x-ct$ (so three minus signs).

$$\begin{aligned}u(x,t)=\frac{1}{2}[(-1)^3\phi(4l-x-ct)+(-1)^4\phi(4l+x-ct)]&\\+\frac{1}{2c}\left[\int_{x-ct}^{-3l}\psi(s+4l)\,ds+\int_{-3l}^{-2l}[-\psi(-s-2l)]\,ds+\int_{-2l}^{-l}\psi(s+2l)\,ds\right.&\\+\int_{-l}^{0}[-\psi(-s)]\,ds+\int_0^l\psi(s)\,ds+\int_l^{2l}[-\psi(-s+2l)]\,ds&\\\left.+\int_{2l}^{3l}\psi(s-2l)\,ds+\int_{3l}^{x+ct}[-\psi(-s+4l)]\,ds\right]&\end{aligned}$$

Substitute $r=s+4l$ in the first integral, $r=-s-2l$ in the second integral, $r=s+2l$ in the third integral, $r=-s$ in the fourth integral, $r=s$ in the fifth integral, $r=-s+2l$ in the sixth integral, $r=s-2l$ in the seventh integral, and $r=-s+4l$ in the eighth integral.

$$\begin{aligned}u(x,t)=\frac{1}{2}[-\phi(4l-x-ct)+\phi(4l+x-ct)]&\\+\frac{1}{2c}\left[\int_{4l+x-ct}^{l}\psi(r)\,dr+\cancel{\int_l^0\psi(r)\,dr}+\cancel{\int_0^l\psi(r)\,dr}\right.&\\+\cancel{\int_l^0\psi(r)\,dr}+\cancel{\int_0^l\psi(r)\,dr}+\cancel{\int_l^0\psi(r)\,dr}&\\\left.+\cancel{\int_0^l\psi(r)\,dr}+\int_l^{4l-x-ct}\psi(r)\,dr\right]&\end{aligned}$$

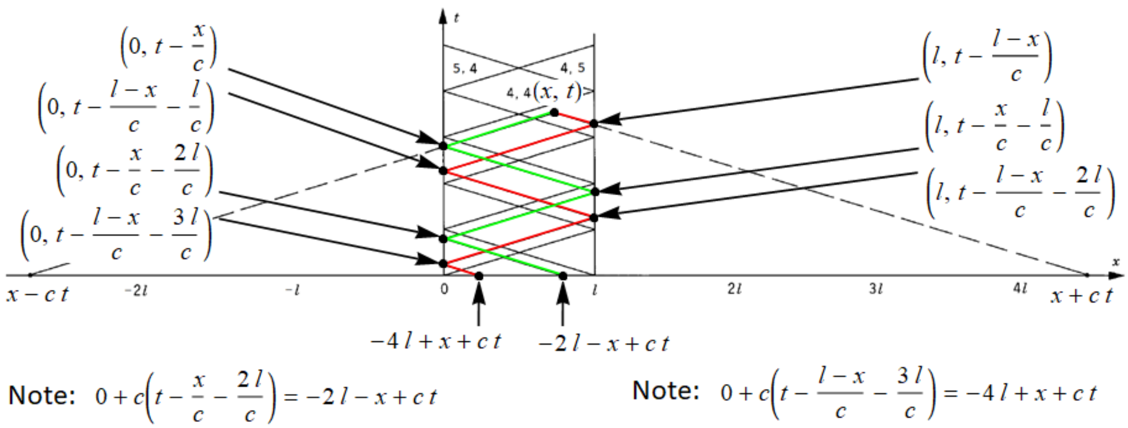
Combine the two remaining integrals.

$$u(x,t) = \frac{1}{2}[-\phi(4l - x - ct) + \phi(4l + x - ct)] + \frac{1}{2c}\int_{4l+x-ct}^{4l-x-ct} \psi(r)\,dr$$

Therefore, the solution in diamond $\langle 4, 3\rangle$ is

$$u(x,t) = \frac{1}{2}[-\phi(4l - x - ct) + \phi(4l + x - ct)] - \frac{1}{2c}\int_{4l-x-ct}^{4l+x-ct} \psi(r)\,dr.$$

Diamond $\langle 3, 4\rangle$



In this diamond reflections occur, so the method of reflection will be used. Consider the corresponding problem over the whole line, using the extensions for ϕ and ψ that are odd with respect to $x = 0$ and $x = l$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned} &v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x,0) = \phi_{\text{ext}}(x) \\ &v_t(x,0) = \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{if } 0 < x < l \\ -\phi(-x) & \text{if } -l < x < 0 \\ \phi_{\text{ext}}(x + 2l) & \end{cases} \quad \text{and} \quad \psi_{\text{ext}}(x) = \begin{cases} \psi(x) & \text{if } 0 < x < l \\ -\psi(-x) & \text{if } -l < x < 0 \\ \psi_{\text{ext}}(x + 2l) & \end{cases}.$$

The solution for v is given by d'Alembert's formula.

$$v(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds$$

u is obtained by restricting this solution to $0 < x < l$.

$$u(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds, \quad 0 < x < l$$

Our task now is to write this formula in terms of the given functions, ϕ and ψ. There are three reflections on the way to $-2l - x + ct$ (so three minus signs), and there are four reflections on the way to $-4l + x + ct$ (so four minus signs).

$$\begin{aligned} u(x,t) = \frac{1}{2}[(-1)^4\phi(-4l+x+ct) + (-1)^3\phi(-2l-x+ct)] \\ + \frac{1}{2c}\left[\int_{x-ct}^{-2l}[-\psi(-s-2l)]\,ds + \int_{-2l}^{-l}\psi(s+2l)\,ds + \int_{-l}^{0}[-\psi(-s)]\,ds \right. \\ + \int_0^l \psi(s)\,ds + \int_l^{2l}[-\psi(-s+2l)]\,ds + \int_{2l}^{3l}\psi(s-2l)\,ds \\ \left. + \int_{3l}^{4l}[-\psi(-s+4l)]\,ds + \int_{4l}^{x+ct}\psi(s-4l)\,ds \right] \end{aligned}$$

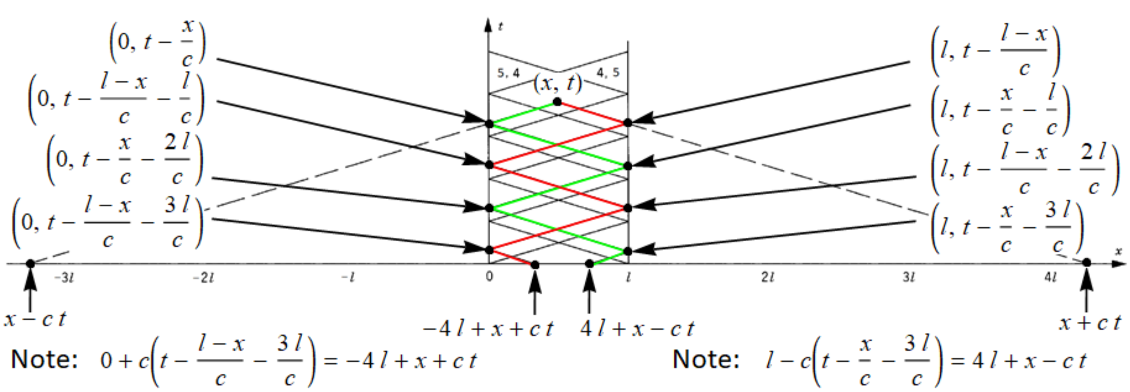
Substitute $r = -s - 2l$ in the first integral, $r = s + 2l$ in the second integral, $r = -s$ in the third integral, $r = s$ in the fourth integral, $r = -s + 2l$ in the fifth integral, $r = s - 2l$ in the sixth integral, $r = -s + 4l$ in the seventh integral, and $r = s - 4l$ in the eighth integral.

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\phi(-4l+x+ct) - \phi(-2l-x+ct)] \\ &\quad + \frac{1}{2c}\left[\int_{-2l-x+ct}^{0}\psi(r)\,dr + \int_0^l \psi(r)\,dr + \int_l^0 \psi(r)\,dr \right. \\ &\quad + \int_0^l \psi(r)\,dr + \int_l^0 \psi(r)\,dr + \int_0^l \psi(r)\,dr \\ &\quad \left. + \int_l^0 \psi(r)\,dr + \int_0^{-4l+x+ct}\psi(r)\,dr \right] \\ &= \frac{1}{2}[\phi(-4l+x+ct) - \phi(-2l-x+ct)] + \frac{1}{2c}\int_{-2l-x+ct}^{-4l+x+ct}\psi(r)\,dr \end{aligned}$$

Therefore, the solution in diamond $\langle 3, 4\rangle$ is

$$u(x,t) = \frac{1}{2}[\phi(-4l+x+ct) - \phi(-2l-x+ct)] - \frac{1}{2c}\int_{-4l+x+ct}^{-2l-x+ct}\psi(r)\,dr.$$

Diamond $\langle 4, 4\rangle$



In this diamond reflections occur, so the method of reflection will be used. Consider the corresponding problem over the whole line, using the extensions for ϕ and ψ that are odd with respect to $x = 0$ and $x = l$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned}
&v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\
&v(x, 0) = \phi_{\text{ext}}(x) \\
&v_t(x, 0) = \psi_{\text{ext}}(x)
\end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{if } 0 < x < l \\ -\phi(-x) & \text{if } -l < x < 0 \\ \phi_{\text{ext}}(x + 2l) \end{cases} \quad \text{and} \quad \psi_{\text{ext}}(x) = \begin{cases} \psi(x) & \text{if } 0 < x < l \\ -\psi(-x) & \text{if } -l < x < 0 \\ \psi_{\text{ext}}(x + 2l) \end{cases}.$$

The solution for v is given by d'Alembert's formula.

$$v(x, t) = \frac{1}{2}[\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds$$

u is obtained by restricting this solution to $0 < x < l$.

$$u(x, t) = \frac{1}{2}[\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds, \quad 0 < x < l$$

Our task now is to write this formula in terms of the given functions, ϕ and ψ. There are four reflections on the way to $4l + x - ct$ (so four minus signs), and there are four reflections on the way to $-4l + x + ct$ (so four minus signs).

$$\begin{aligned}
u(x, t) = \frac{1}{2}&[(-1)^4\phi(-4l + x + ct) + (-1)^4\phi(4l + x - ct)] \\
&+ \frac{1}{2c}\left[\int_{x-ct}^{-3l} \psi(s + 4l)\, ds + \int_{-3l}^{-2l} [-\psi(-s - 2l)]\, ds + \int_{-2l}^{-l} \psi(s + 2l)\, ds \right. \\
&\qquad + \int_{-l}^{0} [-\psi(-s)]\, ds + \int_0^l \psi(s)\, ds + \int_l^{2l} [-\psi(-s + 2l)]\, ds \\
&\qquad \left. + \int_{2l}^{3l} \psi(s - 2l)\, ds + \int_{3l}^{4l} [-\psi(-s + 4l)]\, ds + \int_{4l}^{x+ct} \psi(s - 4l)\, ds \right]
\end{aligned}$$

Substitute $r = s + 4l$ in the first integral, $r = -s - 2l$ in the second integral, $r = s + 2l$ in the third integral, $r = -s$ in the fourth integral, $r = s$ in the fifth integral, $r = -s + 2l$ in the sixth integral, $r = s - 2l$ in the seventh integral, $r = -s + 4l$ in the eighth integral, and $r = s - 4l$ in the ninth integral.

$$\begin{aligned}
u(x, t) = \frac{1}{2}&[\phi(-4l + x + ct) + \phi(4l + x - ct)] \\
&+ \frac{1}{2c}\left[\int_{4l+x-ct}^{l} \psi(r)\, dr + \cancel{\int_l^0 \psi(r)\, dr} + \cancel{\int_0^l \psi(r)\, dr} \right. \\
&\qquad + \cancel{\int_l^0 \psi(r)\, dr} + \cancel{\int_0^l \psi(r)\, dr} + \cancel{\int_l^0 \psi(r)\, dr} \\
&\qquad \left. + \cancel{\int_0^l \psi(r)\, dr} + \int_l^0 \psi(r)\, dr + \int_0^{-4l+x+ct} \psi(r)\, dr \right]
\end{aligned}$$

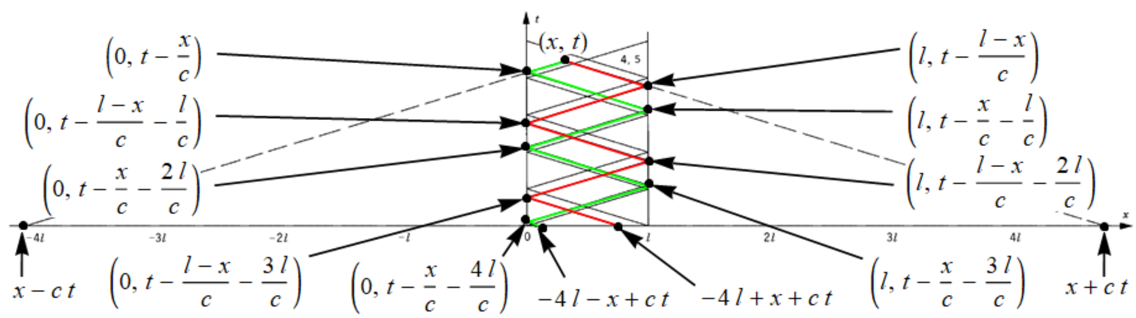
Combine the remaining integrals.

$$u(x,t) = \frac{1}{2}[\phi(-4l + x + ct) + \phi(4l + x - ct)] + \frac{1}{2c}\int_{4l+x-ct}^{-4l+x+ct} \psi(r)\, dr$$

Therefore, the solution in diamond $\langle 4, 4\rangle$ is

$$u(x,t) = \frac{1}{2}[\phi(-4l + x + ct) + \phi(4l + x - ct)] - \frac{1}{2c}\int_{-4l+x+ct}^{4l+x-ct} \psi(r)\, dr.$$

Diamond $\langle 5, 4\rangle$



Note that $0 + c\left(t - \frac{x}{c} - \frac{4l}{c}\right) = -4l - x + ct$ and $0 + c\left(t - \frac{l-x}{c} - \frac{3l}{c}\right) = -4l + x + ct$. In this diamond reflections occur, so the method of reflection will be used. Consider the corresponding problem over the whole line, using the extensions for ϕ and ψ that are odd with respect to $x = 0$ and $x = l$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned} &v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x, 0) = \phi_{\text{ext}}(x) \\ &v_t(x, 0) = \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{if } 0 < x < l \\ -\phi(-x) & \text{if } -l < x < 0 \\ \phi_{\text{ext}}(x + 2l) \end{cases} \quad \text{and} \quad \psi_{\text{ext}}(x) = \begin{cases} \psi(x) & \text{if } 0 < x < l \\ -\psi(-x) & \text{if } -l < x < 0 \\ \psi_{\text{ext}}(x + 2l) \end{cases}.$$

The solution for v is given by d'Alembert's formula.

$$v(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds$$

u is obtained by restricting this solution to $0 < x < l$.

$$u(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds, \quad 0 < x < l$$

Our task now is to write this formula in terms of the given functions, ϕ and ψ. There are five reflections on the way to $-4l - x + ct$ (so five minus signs), and there are four reflections on the way to $-4l + x + ct$ (so four minus signs).

$$\begin{aligned} u(x,t) = \frac{1}{2}[(-1)^4\phi(-4l+x+ct) + (-1)^5\phi(-4l-x+ct)] \\ + \frac{1}{2c}\left[\int_{x-ct}^{-4l}[-\psi(-s-4l)]\,ds + \int_{-4l}^{-3l}\psi(s+4l)\,ds + \int_{-3l}^{-2l}[-\psi(-s-2l)]\,ds \right. \\ + \int_{-2l}^{-l}\psi(s+2l)\,ds + \int_{-l}^{0}[-\psi(-s)]\,ds + \int_{0}^{l}\psi(s)\,ds + \int_{l}^{2l}[-\psi(-s+2l)]\,ds \\ \left. + \int_{2l}^{3l}\psi(s-2l)\,ds + \int_{3l}^{4l}[-\psi(-s+4l)]\,ds + \int_{4l}^{x+ct}\psi(s-4l)\,ds\right] \end{aligned}$$

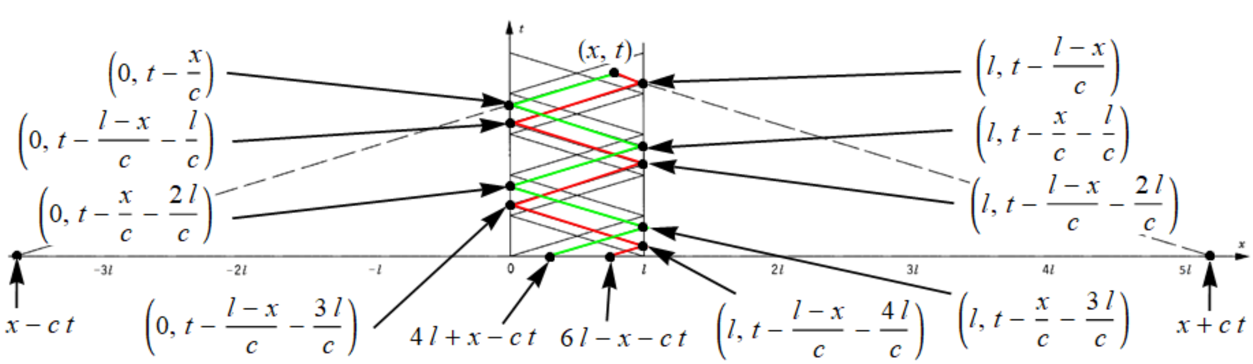
Substitute $r = -s-4l$ in the first integral, $r = s+4l$ in the second integral, $r = -s-2l$ in the third integral, $r = s+2l$ in the fourth integral, $r = -s$ in the fifth integral, $r = s$ in the sixth integral, $r = -s+2l$ in the seventh integral, $r = s-2l$ in the eighth integral, $r = -s+4l$ in the ninth integral, and $r = s-4l$ in the tenth integral.

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\phi(-4l+x+ct) - \phi(-4l-x+ct)] \\ &\quad + \frac{1}{2c}\left[\int_{-4l-x+ct}^{0}\psi(r)\,dr + \int_{0}^{l}\psi(r)\,dr + \cancel{\int_{l}^{0}\psi(r)\,dr} \right. \\ &\quad + \cancel{\int_{0}^{l}\psi(r)\,dr} + \cancel{\int_{l}^{0}\psi(r)\,dr} + \cancel{\int_{0}^{l}\psi(r)\,dr} + \cancel{\int_{l}^{0}\psi(r)\,dr} \\ &\quad \left. + \cancel{\int_{0}^{l}\psi(r)\,dr} + \int_{l}^{0}\psi(r)\,dr + \int_{0}^{-4l+x+ct}\psi(r)\,dr\right] \\ &= \frac{1}{2}[\phi(-4l+x+ct) - \phi(-4l-x+ct)] \\ &\quad + \frac{1}{2c}\left[\int_{-4l-x+ct}^{0}\psi(r)\,dr + \cancel{\int_{0}^{l}\psi(r)\,dr} + \cancel{\int_{l}^{0}\psi(r)\,dr} + \int_{0}^{-4l+x+ct}\psi(r)\,dr\right] \end{aligned}$$

Therefore, the solution in diamond $\langle 5, 4\rangle$ is

$$u(x,t) = \frac{1}{2}[\phi(-4l+x+ct) - \phi(-4l-x+ct)] + \frac{1}{2c}\int_{-4l-x+ct}^{-4l+x+ct}\psi(r)\,dr.$$

Diamond $\langle 4, 5\rangle$



Note that $l - c\left(t - \frac{x}{c} - \frac{3l}{c}\right) = 4l + x - ct$ and $l - c\left(t - \frac{l-x}{c} - \frac{4l}{c}\right) = 6l - x - ct$. In this diamond reflections occur, so the method of reflection will be used. Consider the corresponding problem over the whole line, using the extensions for ϕ and ψ that are odd with respect to $x = 0$ and $x = l$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned} &v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x,0) = \phi_{\text{ext}}(x) \\ &v_t(x,0) = \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{if } 0 < x < l \\ -\phi(-x) & \text{if } -l < x < 0 \\ \phi_{\text{ext}}(x+2l) \end{cases} \quad \text{and} \quad \psi_{\text{ext}}(x) = \begin{cases} \psi(x) & \text{if } 0 < x < l \\ -\psi(-x) & \text{if } -l < x < 0 \\ \psi_{\text{ext}}(x+2l) \end{cases}.$$

The solution for v is given by d'Alembert's formula.

$$v(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds$$

u is obtained by restricting this solution to $0 < x < l$.

$$u(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds, \quad 0 < x < l$$

Our task now is to write this formula in terms of the given functions, ϕ and ψ. There are four reflections on the way to $4l + x - ct$ (so four minus signs), and there are five reflections on the way to $6l - x - ct$ (so five minus signs).

$$\begin{aligned} u(x,t) = \frac{1}{2}&[(-1)^5\phi(6l - x - ct) + (-1)^4\phi(4l + x - ct)] \\ &+ \frac{1}{2c}\left[\int_{x-ct}^{-3l} \psi(s+4l)\,ds + \int_{-3l}^{-2l} [-\psi(-s-2l)]\,ds + \int_{-2l}^{-l} \psi(s+2l)\,ds \right. \\ &\qquad + \int_{-l}^{0} [-\psi(-s)]\,ds + \int_0^l \psi(s)\,ds + \int_l^{2l} [-\psi(-s+2l)]\,ds + \int_{2l}^{3l} \psi(s-2l)\,ds \\ &\qquad \left. + \int_{3l}^{4l} [-\psi(-s+4l)]\,ds + \int_{4l}^{5l} \psi(s-4l)\,ds + \int_{5l}^{x+ct} [-\psi(-s+6l)]\,ds \right] \end{aligned}$$

Substitute $r = s + 4l$ in the first integral, $r = -s - 2l$ in the second integral, $r = s + 2l$ in the third integral, $r = -s$ in the fourth integral, $r = s$ in the fifth integral, $r = -s + 2l$ in the sixth integral, $r = s - 2l$ in the seventh integral, $r = -s + 4l$ in the eighth integral, $r = s - 4l$ in the ninth integral, and $r = -s + 6l$ in the tenth integral.

$$\begin{aligned} u(x,t) = \frac{1}{2}&[-\phi(6l - x - ct) + \phi(4l + x - ct)] \\ &+ \frac{1}{2c}\left[\int_{4l+x-ct}^{l} \psi(r)\,dr + \cancel{\int_l^0 \psi(r)\,dr} + \cancel{\int_0^l \psi(r)\,dr} \right. \\ &\qquad + \cancel{\int_l^0 \psi(r)\,dr} + \cancel{\int_0^l \psi(r)\,dr} + \cancel{\int_l^0 \psi(r)\,dr} + \cancel{\int_0^l \psi(r)\,dr} \\ &\qquad \left. + \int_l^0 \psi(r)\,dr + \int_0^l \psi(r)\,dr + \int_l^{6l-x-ct} \psi(r)\,dr \right] \end{aligned}$$

Combine the remaining integrals.

$$u(x,t) = \frac{1}{2}[-\phi(6l - x - ct) + \phi(4l + x - ct)]$$
$$+ \frac{1}{2c}\left[\int_{4l+x-ct}^{l} \psi(r)\,dr + \cancel{\int_{l}^{0} \psi(r)\,dr} + \cancel{\int_{0}^{l} \psi(r)\,dr} + \int_{l}^{6l-x-ct} \psi(r)\,dr\right]$$

Therefore, the solution in diamond $\langle 4,5\rangle$ is

$$u(x,t) = \frac{1}{2}[-\phi(6l - x - ct) + \phi(4l + x - ct)] + \frac{1}{2c}\int_{4l+x-ct}^{6l-x-ct} \psi(r)\,dr.$$

In conclusion,

$$u(x,t) = \begin{cases}
\frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(r)\,dr & \text{in } \langle 0,0\rangle \\
\frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(r)\,dr & \text{in } \langle 1,0\rangle \\
\frac{1}{2}[-\phi(2l-x-ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{2l-x-ct} \psi(r)\,dr & \text{in } \langle 0,1\rangle \\
-\frac{1}{2}[\phi(2l-x-ct) + \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{2l-x-ct} \psi(r)\,dr & \text{in } \langle 1,1\rangle \\
\frac{1}{2}[-\phi(2l-x-ct) + \phi(2l+x-ct)] - \int_{2l-x-ct}^{2l+x-ct} \psi(r)\,dr & \text{in } \langle 2,1\rangle \\
\frac{1}{2}[\phi(-2l+x+ct) - \phi(ct-x)] - \int_{-2l+x+ct}^{ct-x} \psi(r)\,dr & \text{in } \langle 1,2\rangle \\
\frac{1}{2}[\phi(-2l+x+ct) + \phi(2l+x-ct)] - \frac{1}{2c}\int_{-2l+x+ct}^{2l+x-ct} \psi(r)\,dr & \text{in } \langle 2,2\rangle \\
\frac{1}{2}[\phi(-2l+x+ct) - \phi(-2l-x+ct)] + \int_{-2l-x+ct}^{-2l+x+ct} \psi(r)\,dr & \text{in } \langle 3,2\rangle \\
\frac{1}{2}[-\phi(4l-x-ct) + \phi(2l+x-ct)] + \frac{1}{2c}\int_{2l+x-ct}^{4l-x-ct} \psi(r)\,dr & \text{in } \langle 2,3\rangle \\
-\frac{1}{2}[\phi(4l-x-ct) + \phi(-2l-x+ct)] + \frac{1}{2c}\int_{-2l-x+ct}^{4l-x-ct} \psi(r)\,dr & \text{in } \langle 3,3\rangle \\
\frac{1}{2}[-\phi(4l-x-ct) + \phi(4l+x-ct)] - \frac{1}{2c}\int_{4l-x-ct}^{4l+x-ct} \psi(r)\,dr & \text{in } \langle 4,3\rangle \\
\frac{1}{2}[\phi(-4l+x+ct) - \phi(-2l-x+ct)] - \frac{1}{2c}\int_{-4l+x+ct}^{-2l-x+ct} \psi(r)\,dr & \text{in } \langle 3,4\rangle \\
\frac{1}{2}[\phi(-4l+x+ct) + \phi(4l+x-ct)] - \frac{1}{2c}\int_{-4l+x+ct}^{4l+x-ct} \psi(r)\,dr & \text{in } \langle 4,4\rangle \\
\frac{1}{2}[\phi(-4l+x+ct) - \phi(-4l-x+ct)] + \frac{1}{2c}\int_{-4l-x+ct}^{-4l+x+ct} \psi(r)\,dr & \text{in } \langle 5,4\rangle \\
\frac{1}{2}[-\phi(6l-x-ct) + \phi(4l+x-ct)] + \frac{1}{2c}\int_{4l+x-ct}^{6l-x-ct} \psi(r)\,dr & \text{in } \langle 4,5\rangle
\end{cases}.$$