Exercise 11

Solve $u_{tt} = c^2 u_{xx}$ in $0 < x < l$, $u(x,0) = 0$, $u_t(x,0) = x$, $u(0,t) = u(l,t) = 0$.

Solution

The method of reflection will be used to solve this problem. Consider the corresponding problem over the whole line, using the extensions for the initial data that are odd with respect to $x = 0$ and $x = l$ in order to satisfy the homogeneous boundary conditions.

$$\begin{aligned} &v_{tt} = c^2 v_{xx} \qquad -\infty < x < \infty,\ t > 0 \\ &v(x,0) = \phi_{\text{ext}}(x) \\ &v_t(x,0) = \psi_{\text{ext}}(x) \end{aligned}$$

Here $\phi_{\text{ext}}(x)$ and $\psi_{\text{ext}}(x)$ are defined as

$$\phi_{\text{ext}}(x) = 0 \quad \text{and} \quad \psi_{\text{ext}}(x) = \begin{cases} x & \text{if } 0 < x < l \\ -(-x) & \text{if } -l < x < 0 \\ \psi_{\text{ext}}(x + 2l) \end{cases}.$$

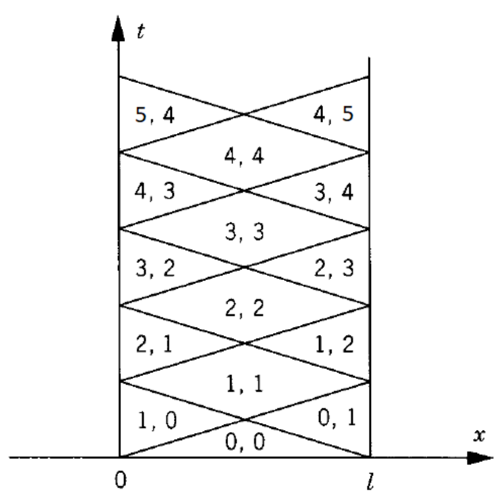
The solution for v is given by d'Alembert's formula.

$$\begin{aligned} v(x,t) &= \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds \end{aligned}$$

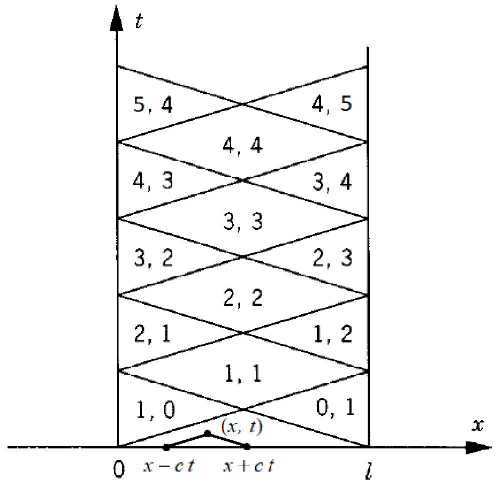
u is obtained by restricting this solution to $0 < x < l$.

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds, \quad 0 < x < l$$

Depending what point (x,t) is chosen, the integral will yield a different result. We will calculate it for points within each of the diamond-shaped regions illustrated below.

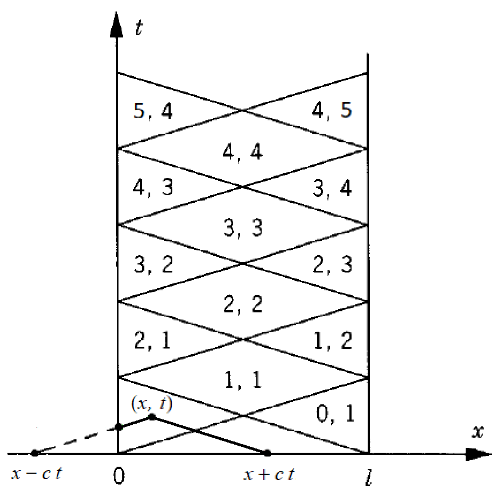


Diamond $\langle 0, 0 \rangle$



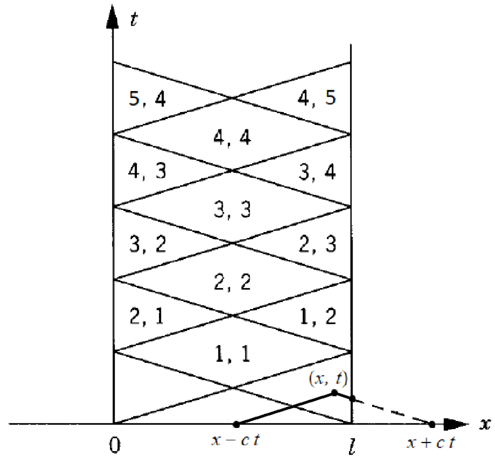
$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} s\,ds \\ &= \frac{1}{2c}(2cxt) \\ &= xt \end{aligned}$$

Diamond $\langle 1, 0 \rangle$



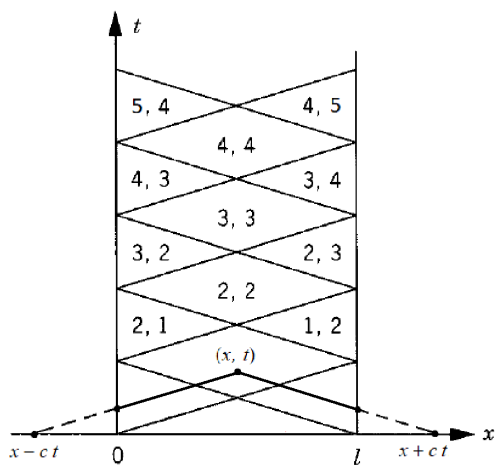
$$\begin{aligned} u(x,t) &= \frac{1}{2c}\left[\int_{x-ct}^{0} [-(-s)]\,ds + \int_{0}^{x+ct} s\,ds\right] \\ &= \frac{1}{2c}(2cxt) \\ &= xt \end{aligned}$$

Diamond $\langle 0, 1\rangle$



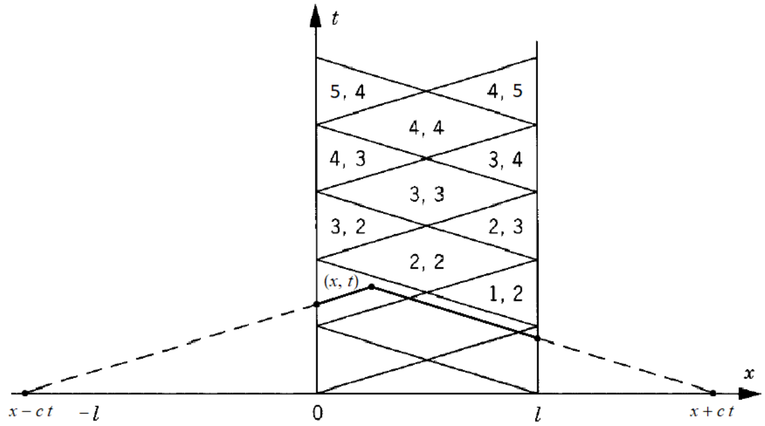
$$u(x,t) = \frac{1}{2c}\left[\int_{x-ct}^{l} s\,ds + \int_{l}^{x+ct} [-(-s+2l)]\,ds\right]$$
$$= \frac{1}{2c}[2(l-ct)(l-x)]$$
$$= \frac{(l-ct)(l-x)}{c}$$

Diamond $\langle 1, 1\rangle$



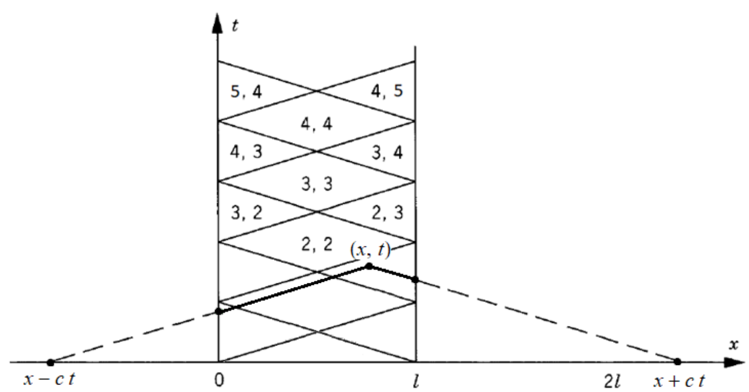
$$u(x,t) = \frac{1}{2c}\left[\int_{x-ct}^{0} [-(-s)]\,ds + \int_{0}^{l} s\,ds + \int_{l}^{x+ct} [-(-s+2l)]\,ds\right]$$
$$= \frac{1}{2c}[2(l-ct)(l-x)]$$
$$= \frac{(l-ct)(l-x)}{c}$$

Diamond $\langle 2, 1\rangle$



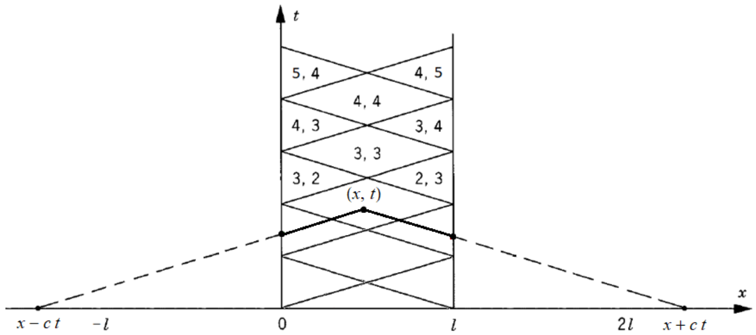
$$\begin{aligned} u(x,t) &= \frac{1}{2c}\left[\int_{x-ct}^{-l}(s+2l)\,ds + \int_{-l}^{0}[-(-s)]\,ds + \int_0^l s\,ds + \int_l^{x+ct}[-(-s+2l)]\,ds\right] \\ &= \frac{1}{2c}[-2x(2l-ct)] \\ &= -\frac{x(2l-ct)}{c} \end{aligned}$$

Diamond $\langle 1, 2\rangle$



$$\begin{aligned} u(x,t) &= \frac{1}{2c}\left[\int_{x-ct}^{0}[-(-s)]\,ds + \int_0^l s\,ds + \int_l^{2l}[-(-s+2l)]\,ds + \int_{2l}^{x+ct}(s-2l)\,ds\right] \\ &= \frac{1}{2c}[2(l-ct)(l-x)] \\ &= \frac{(l-ct)(l-x)}{c} \end{aligned}$$

Diamond $\langle 2, 2\rangle$

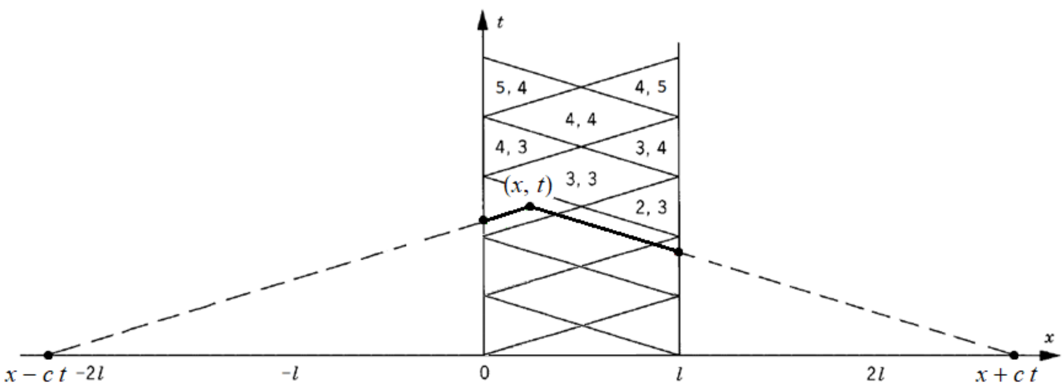


$$u(x,t) = \frac{1}{2c}\left[\int_{x-ct}^{-l}(s+2l)\,ds + \int_{-l}^{0}[-(-s)]\,ds + \int_{0}^{l} s\,ds + \int_{l}^{2l}[-(-s+2l)]\,ds + \int_{2l}^{x+ct}(s-2l)\,ds\right]$$

$$= \frac{1}{2c}[-2x(2l-ct)]$$

$$= -\frac{x(2l-ct)}{c}$$

Diamond $\langle 3, 2\rangle$

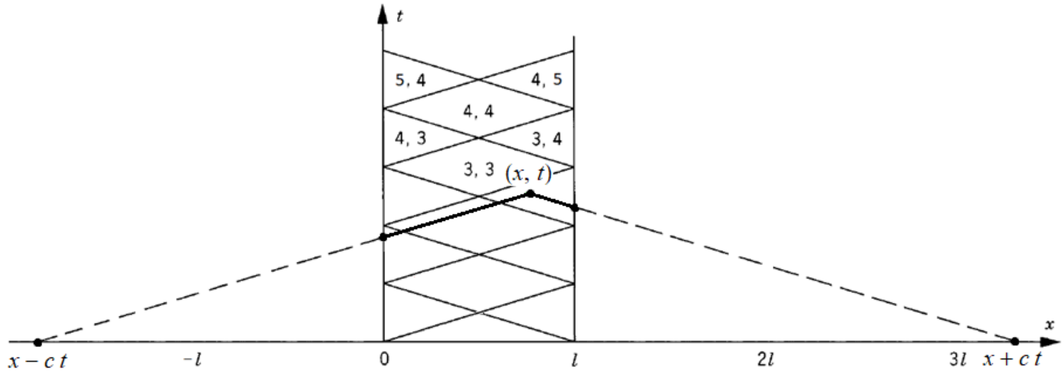


$$u(x,t) = \frac{1}{2c}\left[\int_{x-ct}^{-2l}[-(-s-2l)]\,ds + \int_{-2l}^{-l}(s+2l)\,ds + \int_{-l}^{0}[-(-s)]\,ds + \int_{0}^{l} s\,ds + \int_{l}^{2l}[-(-s+2l)]\,ds + \int_{2l}^{x+ct}(s-2l)\,ds\right]$$

$$= \frac{1}{2c}[-2x(2l-ct)]$$

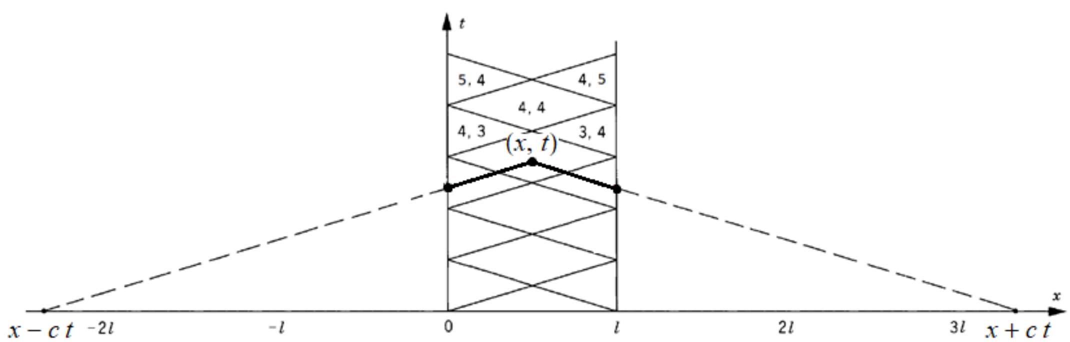
$$= -\frac{x(2l-ct)}{c}$$

Diamond $\langle 2, 3\rangle$



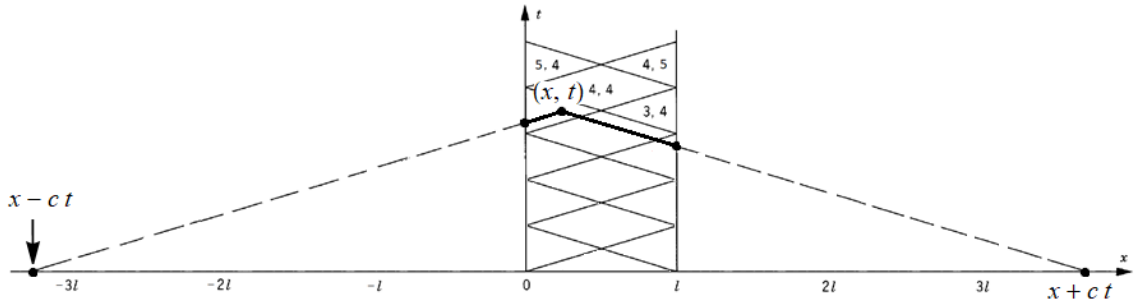
$$
\begin{aligned}
u(x,t) &= \frac{1}{2c}\left[\int_{x-ct}^{-l}(s+2l)\,ds + \int_{-l}^{0}[-(-s)]\,ds + \int_{0}^{l} s\,ds \right.\\
&\qquad\left. + \int_{l}^{2l}[-(-s+2l)]\,ds + \int_{2l}^{3l}(s-2l)\,ds + \int_{3l}^{x+ct}[-(-s+4l)]\,ds\right]\\
&= \frac{1}{2c}[2(3l-ct)(l-x)]\\
&= \frac{(3l-ct)(l-x)}{c}
\end{aligned}
$$

Diamond $\langle 3, 3\rangle$



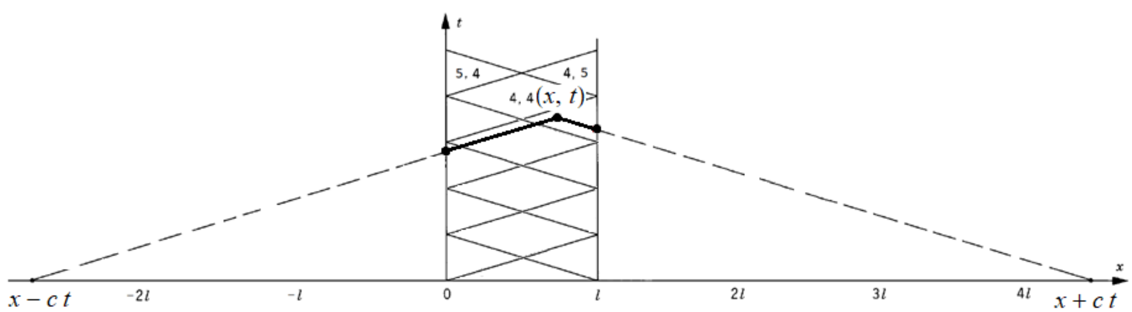
$$
\begin{aligned}
u(x,t) &= \frac{1}{2c}\left[\int_{x-ct}^{-2l}[-(-s-2l)]\,ds + \int_{-2l}^{-l}(s+2l)\,ds + \int_{-l}^{0}[-(-s)]\,ds + \int_{0}^{l} s\,ds \right.\\
&\qquad\left. + \int_{l}^{2l}[-(-s+2l)]\,ds + \int_{2l}^{3l}(s-2l)\,ds + \int_{3l}^{x+ct}[-(-s+4l)]\,ds\right]\\
&= \frac{1}{2c}[2(3l-ct)(l-x)]\\
&= \frac{(3l-ct)(l-x)}{c}
\end{aligned}
$$

Diamond $\langle 4, 3\rangle$



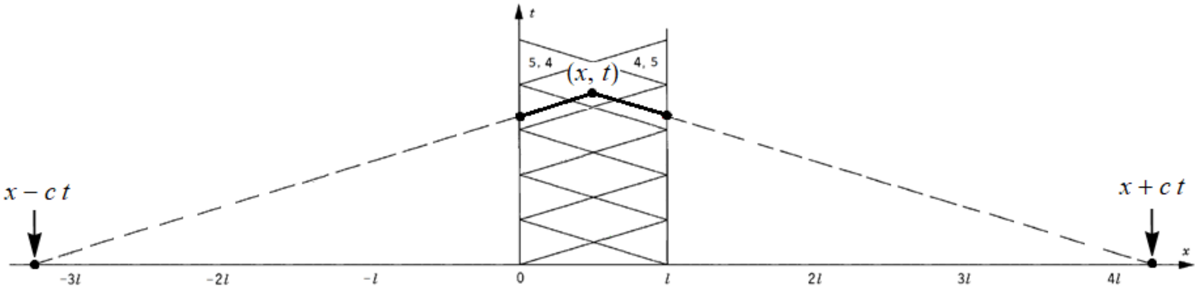
$$u(x,t) = \frac{1}{2c}\left[\int_{x-ct}^{-3l}(s+4l)\,ds + \int_{-3l}^{-2l}[-(-s-2l)]\,ds + \int_{-2l}^{-l}(s+2l)\,ds \right.$$
$$+ \int_{-l}^{0}[-(-s)]\,ds + \int_{0}^{l} s\,ds + \int_{l}^{2l}[-(-s+2l)]\,ds$$
$$\left. + \int_{2l}^{3l}(s-2l)\,ds + \int_{3l}^{x+ct}[-(-s+4l)]\,ds\right]$$
$$= \frac{1}{2c}[-2x(4l-ct)]$$
$$= -\frac{x(4l-ct)}{c}$$

Diamond $\langle 3, 4\rangle$



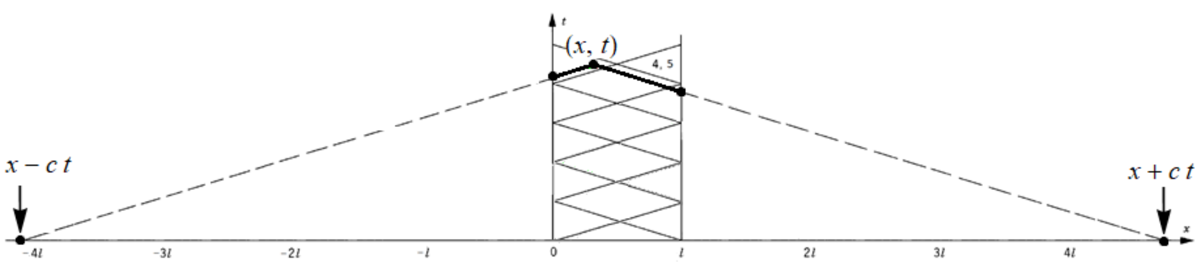
$$u(x,t) = \frac{1}{2c}\left[\int_{x-ct}^{-2l}[-(-s-2l)]\,ds + \int_{-2l}^{-l}(s+2l)\,ds + \int_{-l}^{0}[-(-s)]\,ds \right.$$
$$+ \int_{0}^{l} s\,ds + \int_{l}^{2l}[-(-s+2l)]\,ds + \int_{2l}^{3l}(s-2l)\,ds$$
$$\left. + \int_{3l}^{4l}[-(-s+4l)]\,ds + \int_{4l}^{x+ct}(s-4l)\,ds\right]$$
$$= \frac{1}{2c}[2(3l-ct)(l-x)]$$
$$= \frac{(3l-ct)(l-x)}{c}$$

Diamond $\langle 4, 4\rangle$



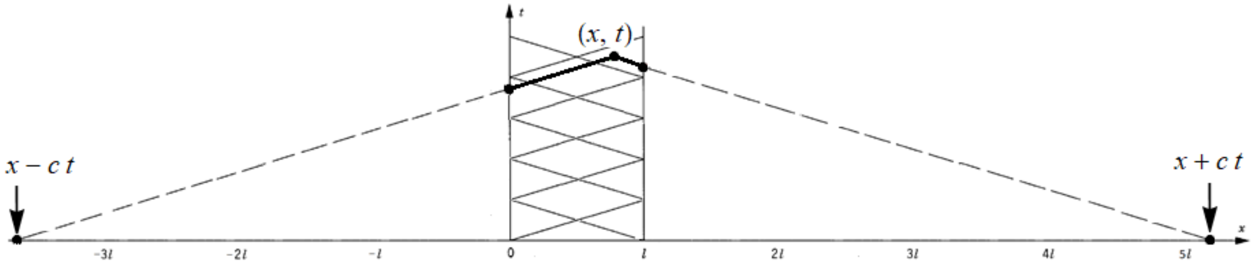
$$\begin{aligned}
u(x,t) = \frac{1}{2c}\left[\int_{x-ct}^{-3l}(s+4l)\,ds + \int_{-3l}^{-2l}[-(-s-2l)]\,ds + \int_{-2l}^{-l}(s+2l)\,ds\right.\\
+ \int_{-l}^{0}[-(-s)]\,ds + \int_{0}^{l} s\,ds + \int_{l}^{2l}[-(-s+2l)]\,ds\\
\left. + \int_{2l}^{3l}(s-2l)\,ds + \int_{3l}^{4l}[-(-s+4l)]\,ds + \int_{4l}^{x+ct}(s-4l)\,ds\right]\\
= \frac{1}{2c}[-2x(4l-ct)]\\
= -\frac{x(4l-ct)}{c}
\end{aligned}$$

Diamond $\langle 5, 4\rangle$



$$\begin{aligned}
u(x,t) = \frac{1}{2c}\left[\int_{x-ct}^{-4l}[-(-s-4l)]\,ds + \int_{-4l}^{-3l}(s+4l)\,ds + \int_{-3l}^{-2l}[-(-s-2l)]\,ds\right.\\
+ \int_{-2l}^{-l}(s+2l)\,ds + \int_{-l}^{0}[-(-s)]\,ds + \int_{0}^{l} s\,ds + \int_{l}^{2l}[-(-s+2l)]\,ds\\
\left. + \int_{2l}^{3l}(s-2l)\,ds + \int_{3l}^{4l}[-(-s+4l)]\,ds + \int_{4l}^{x+ct}(s-4l)\,ds\right]\\
= \frac{1}{2c}[-2x(4l-ct)]\\
= -\frac{x(4l-ct)}{c}
\end{aligned}$$

Diamond $\langle 4, 5\rangle$



$$\begin{aligned}
u(x,t) &= \frac{1}{2c}\left[\int_{x-ct}^{-3l}(s+4l)\,ds + \int_{-3l}^{-2l}[-(-s-2l)]\,ds + \int_{-2l}^{-l}(s+2l)\,ds\right.\\
&\qquad + \int_{-l}^{0}[-(-s)]\,ds + \int_{0}^{l} s\,ds + \int_{l}^{2l}[-(-s+2l)]\,ds + \int_{2l}^{3l}(s-2l)\,ds\\
&\qquad\left. + \int_{3l}^{4l}[-(-s+4l)]\,ds + \int_{4l}^{5l}(s-4l)\,ds + \int_{5l}^{x+ct}[-(-s+6l)]\,ds\right]\\
&= \frac{1}{2c}[2(5l-ct)(l-x)]\\
&= \frac{(5l-ct)(l-x)}{c}
\end{aligned}$$

Therefore,

$$u(x,t) = \begin{cases} xt & \text{in } \langle 0,0\rangle \\ xt & \text{in } \langle 1,0\rangle \\ \frac{(l-ct)(l-x)}{c} & \text{in } \langle 0,1\rangle \\ \frac{(l-ct)(l-x)}{c} & \text{in } \langle 1,1\rangle \\ -\frac{x(2l-ct)}{c} & \text{in } \langle 2,1\rangle \\ \frac{(l-ct)(l-x)}{c} & \text{in } \langle 1,2\rangle \\ -\frac{x(2l-ct)}{c} & \text{in } \langle 2,2\rangle \\ -\frac{x(2l-ct)}{c} & \text{in } \langle 3,2\rangle \\ \frac{(3l-ct)(l-x)}{c} & \text{in } \langle 2,3\rangle \\ \frac{(3l-ct)(l-x)}{c} & \text{in } \langle 3,3\rangle \\ -\frac{x(4l-ct)}{c} & \text{in } \langle 4,3\rangle \\ \frac{(3l-ct)(l-x)}{c} & \text{in } \langle 3,4\rangle \\ -\frac{x(4l-ct)}{c} & \text{in } \langle 4,4\rangle \\ -\frac{x(4l-ct)}{c} & \text{in } \langle 5,4\rangle \\ \frac{(5l-ct)(l-x)}{c} & \text{in } \langle 4,5\rangle \\ \vdots & \vdots \end{cases}.$$

There are a few things one should know about this solution. First, each formula satisfies the wave equation. Secondly, the formula for the one region in contact with $t = 0$, diamond $\langle 0,0\rangle$, satisfies the initial conditions, $u(x,0) = 0$ and $u_t(x,0) = x$. Thirdly, the formulas for all of the regions in contact with $x = 0$, namely diamonds $\langle 1,0\rangle$, $\langle 2,1\rangle$, $\langle 3,2\rangle$, $\langle 4,3\rangle$, and $\langle 5,4\rangle$, satisfy the boundary condition $u(0,t) = 0$. Finally, the formulas for all of the regions in contact with $x = l$, namely diamonds $\langle 0,1\rangle$, $\langle 1,2\rangle$, $\langle 2,3\rangle$, $\langle 3,4\rangle$, and $\langle 4,5\rangle$, satisfy the boundary condition $u(l,t) = 0$.