Exercise 4

Repeat Exercise 3 if the end is free.

Solution

The governing equation of motion for a homogeneous string is the wave equation. If the end of the string at $x = 0$ is free, then there is a Neumann boundary condition at $x = 0$. The two initial conditions are obtained from the pre-existing wave $f(x + ct)$—one by setting $t = 0$ and the second by differentiating with respect to t and then setting $t = 0$. Consequently, the initial boundary value problem to solve is

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty,\ t > 0$$
$$u(x, 0) = f(x) \quad u_t(x, 0) = cf'(x)$$
$$u_x(0, t) = 0.$$

Since we're interested in the solution on $0 < x < \infty$, the method of reflection can be applied to solve the PDE. Consider the same problem over the whole line, where the even extensions of the given functions are used in order to satisfy the Neumann boundary condition at $x = 0$.

$$v_{tt} = c^2 v_{xx}, \quad -\infty < x < \infty,\ t > 0$$
$$v(x, 0) = f_{\text{even}}(x), \quad v_t(x, 0) = cf'_{\text{even}}(x),$$

where

$$f_{\text{even}}(x) = \begin{cases} f(x) & \text{if } x > 0 \\ f(-x) & \text{if } x < 0 \end{cases} \quad \text{and} \quad cf'_{\text{even}}(x) = \begin{cases} cf'(x) & \text{if } x > 0 \\ cf'(-x) & \text{if } x < 0 \end{cases}.$$

The solution for v is given by d'Alembert's formula in section 2.1 on page 36.

$$v(x, t) = \frac{1}{2}[f_{\text{even}}(x + ct) + f_{\text{even}}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'_{\text{even}}(s)\, ds$$

The solution for u is then just the restriction of v to $x > 0$.

$$u(x, t) = \frac{1}{2}[f_{\text{even}}(x + ct) + f_{\text{even}}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'_{\text{even}}(s)\, ds, \quad x > 0$$

Our task now is to write this formula in terms of the given function f. Note that

$$f_{\text{even}}(x + ct) = \begin{cases} f(x + ct) & \text{if } x + ct > 0 \\ f(-x - ct) & \text{if } x + ct < 0 \end{cases} \quad \text{and} \quad f_{\text{even}}(x - ct) = \begin{cases} f(x - ct) & \text{if } x - ct > 0 \\ f(-x + ct) & \text{if } x - ct < 0 \end{cases},$$

so for every region in the xt-quarter-plane, we have to test whether $x - ct$ and $x + ct$ are greater than or less than zero. The characteristic curve $x - ct = 0$ is the line that separates the regions. They are illustrated below in Figure 1.

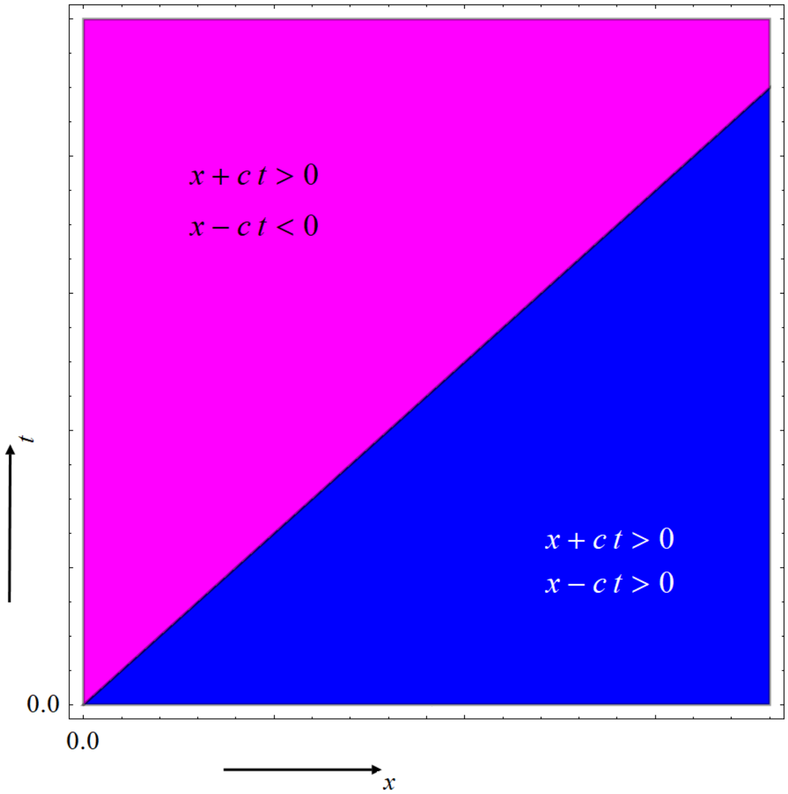

Figure 1: This figure illustrates the regions in the xt-quarter-plane that come about from using the even extensions of f and cf'. The solution for u has to be considered in each one. The characteristic line $x - ct = 0$ is the line that separates the regions.

The Magenta Region

In the magenta region $x + ct > 0$ and $x - ct < 0$, so the solution for u is

$$\begin{aligned}
u(x,t) &= \frac{1}{2}[f_{\text{even}}(x+ct) + f_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'_{\text{even}}(s)\,ds \\
&= \frac{1}{2}[f(x+ct) + f(-x+ct)] + \frac{1}{2c}\left\{\int_{x-ct}^{0} cf'(-s)\,ds + \int_0^{x+ct} cf'(s)\,ds\right\}.
\end{aligned}$$

Cancel c and make the substitution $q = -s$ and $dq = -ds$ in the first integral.

$$\begin{aligned}
&= \frac{1}{2}[f(x+ct) + f(-x+ct)] + \frac{1}{2}\left[\int_{-x+ct}^{0} f'(q)(-dq) + \int_0^{x+ct} f'(s)\,ds\right] \\
&= \frac{1}{2}[f(x+ct) + f(-x+ct)] + \frac{1}{2}\left[\int_0^{-x+ct} f'(q)\,dq + \int_0^{x+ct} f'(s)\,ds\right] \\
&= \frac{1}{2}[f(x+ct) + f(-x+ct)] + \frac{1}{2}\left[f(-x+ct) - f(0) + f(x+ct) - f(0)\right] \\
&= f(x+ct) + f(-x+ct) - f(0)
\end{aligned}$$

The Blue Region

In the blue region $x + ct > 0$ and $x - ct > 0$, so the solution for u is

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f_{\text{even}}(x+ct) + f_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'_{\text{even}}(s)\,ds \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} cf'(s)\,ds \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2}f(s)\Big|_{x-ct}^{x+ct} \\ &= \frac{1}{2}[f(x+ct) + \cancel{f(x-ct)}] + \frac{1}{2}[f(x+ct) - \cancel{f(x-ct)}] \\ &= f(x+ct). \end{aligned}$$

Combining the two solutions, we have

$$u(x,t) = \begin{cases} f(x+ct) + f(-x+ct) - f(0) & \text{if } x - ct < 0 \\ f(x+ct) & \text{if } x - ct > 0 \end{cases}.$$

Note that if $x = 0$, then the $x - ct < 0$ condition applies, and $u_x(0,t) = f(ct) - f(ct) = 0$. The Neumann boundary condition is satisfied; because of this, any constant C can be added to the solution, and the result will still satisfy the wave equation and the boundary condition.

$$u(x,t) = f(x+ct) + f(-x+ct) - f(0) + C, \quad x - ct < 0$$

C can be conveniently chosen so that all the constants in the solution sum to zero.

$$-f(0) + C = 0$$

Therefore,

$$u(x,t) = \begin{cases} f(x+ct) + f(-x+ct) & \text{if } x - ct < 0 \\ f(x+ct) & \text{if } x - ct > 0 \end{cases}.$$

Physical Significance of the Solution

To understand the signficance of the formula for u, it's best to consider an example. Suppose the pre-existing waveform is a Gaussian pulse

$$f(s) = e^{-5(s-1)^2}$$

with speed $c = 1$ so that

$$u(x,t) = \begin{cases} e^{-5(x+t-1)^2} + e^{-5(-x+t-1)^2} & \text{if } x - t < 0 \\ e^{-5(x+t-1)^2} & \text{if } x - t > 0 \\ e^{-5(x+t-1)^2} & \text{if } t < 0 \end{cases}.$$

Graphing u versus x for various times of t will give us insight into the physics of a string with a free end.

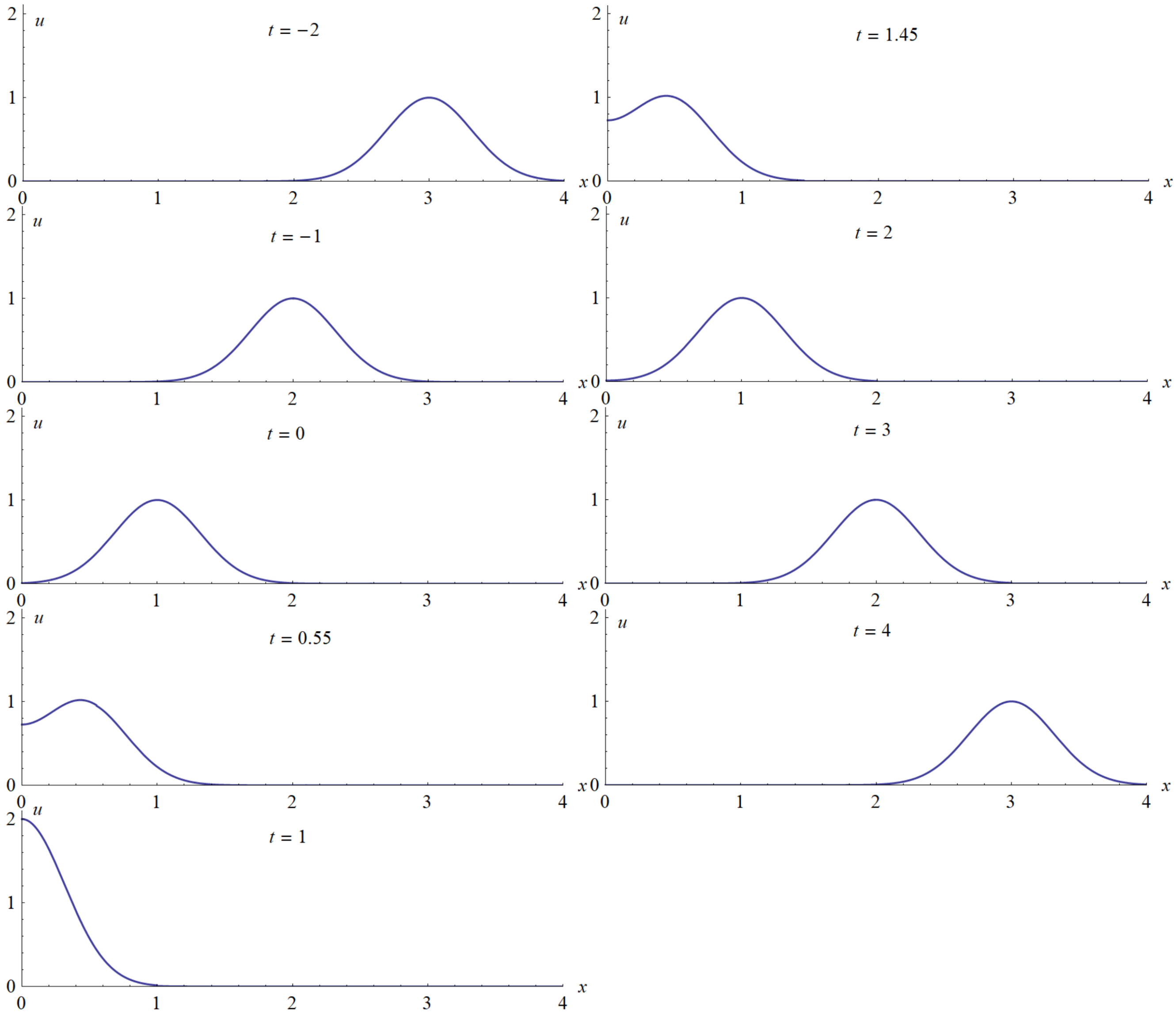

Figure 2: This figure illustrates the time evolution of the sample Gaussian pulse from 2 seconds into the past to 4 seconds into the future. This can be interpreted physically as the motion of a homogeneous elastic string that is free to move at the left end. The propagating wave doubles in amplitude for a moment once it reaches the boundary and travels back with the same speed to where it came from. No inversion takes place as in the case with a fixed end.

The solution for u on the whole line is essentially the sum of two waves, one Gaussian pulse travelling from right to left in the upper half of the xu-plane and one Gaussian pulse travelling from left to right in the upper half of the xu-plane. Once the tails of the curves meet at $t = 0$ and $x = 0$, the waves begin to superimpose and interfere constructively with one another. At $t = 1$ the waves are exactly on top of each other and effectively add together. The wave we see for $t > 2$ is the wave that came from the left of $x = 0$ during $t < 0$.