Exercise 5

Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty$, $u(0,t) = 0$, $u(x,0) \equiv 1$, $u_t(x,0) \equiv 0$ using the reflection method. This solution has a singularity; find its location.

Solution

Since we're interested in the solution on $0 < x < \infty$, the method of reflection can be applied to solve the PDE. Consider the same problem over the whole line, where the odd extension of $u(x,0)$ is used in order to satisfy the Dirichlet boundary condition at $x = 0$.

$$v_{tt} = 4v_{xx}, \quad -\infty < x < \infty, \ t > 0$$
$$v(x,0) = \phi_{\text{odd}}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}, \quad v_t(x,0) = 0$$

The solution for v is given by d'Alembert's formula in section 2.1 on page 36.

$$v(x,t) = \frac{1}{2}[\phi_{\text{odd}}(x+2t) + \phi_{\text{odd}}(x-2t)]$$

The solution for u is then just the restriction of v to $x > 0$.

$$u(x,t) = \frac{1}{2}[\phi_{\text{odd}}(x+2t) + \phi_{\text{odd}}(x-2t)], \quad x > 0$$

Our task now is to write this formula in terms of the given function for $u(x,0)$. Note that

$$\phi_{\text{odd}}(x+2t) = \begin{cases} 1 & \text{if } x+2t > 0 \\ -1 & \text{if } x+2t < 0 \end{cases} \quad \text{and} \quad \phi_{\text{odd}}(x-ct) = \begin{cases} 1 & \text{if } x-2t > 0 \\ -1 & \text{if } x-2t < 0 \end{cases},$$

so for every region in the xt-quarter-plane, we have to test whether $x - 2t$ and $x + 2t$ are greater than or less than zero. The characteristic curve $x - 2t = 0$ is the line that separates the regions. They are illustrated below in Figure 1.

The Magenta Region

In the magenta region $x + 2t > 0$ and $x - 2t < 0$, so the solution for u is

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\phi_{\text{odd}}(x+2t) + \phi_{\text{odd}}(x-2t)] \\ &= \frac{1}{2}[1 + (-1)] = 0. \end{aligned}$$

The Blue Region

In the blue region $x + 2t > 0$ and $x - 2t > 0$, so the solution for u is

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)] \\ &= \frac{1}{2}(1+1) = 1. \end{aligned}$$

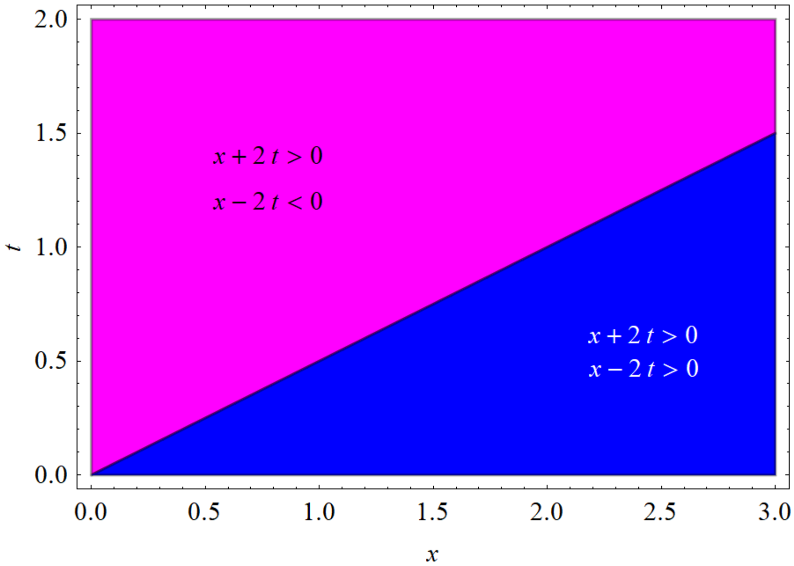

Figure 1: This figure illustrates the regions in the xt-quarter-plane that come about from using the odd extension of $u(x, 0) = 1$. The solution for u has to be considered in each one. The characteristic line $x - 2t = 0$ is the line that separates the regions.

Therefore,

$$u(x,t) = \begin{cases} 0 & \text{if } x - 2t < 0 \\ 1 & \text{if } x - 2t > 0 \end{cases}.$$

A singularity occurs where the solution is discontinuous, that is, at $x - 2t = 0$. If $x = 0$, then the $x - 2t < 0$ condition applies, and $u(0, t) = 0$. The Dirichlet boundary condition is satisfied. In addition, if $t = 0$, then the $x - 2t > 0$ condition applies, and $u(x, 0) = 1$ and $u_t(x, 0) = 0$. The initial conditions are satisfied as well.