Exercise 1

Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \qquad (0 < x < \infty, \quad 0 < t < \infty) \\ u(0,t) &= 0 \qquad u(x,0) = \phi(x) \end{aligned}$$

using the method of reflection.

Solution

Solution by the Method of Reflection

The strategy here is to solve the same problem over the whole line ($-\infty < x < \infty$) using the odd extensions of the given functions, $\phi(x)$ and $f(x,t)$, and then to restrict the domain to $x > 0$ in order to get the desired solution over the half-line. The point of using the odd extensions of the functions is so that the Dirichlet boundary condition $u(0,t) = 0$ is satisfied. Let $U(x,t)$ be the solution to the same problem over the whole line,

$$\begin{gathered} U_t - kU_{xx} = f_{\text{odd}}(x,t), \quad -\infty < x < \infty, \ t > 0 \\ U(x,0) = \phi_{\text{odd}}(x), \end{gathered}$$

where

$$f_{\text{odd}}(x,t) = \begin{cases} f(x,t) & x > 0 \\ -f(-x,t) & x < 0 \end{cases} \quad \text{and} \quad \phi_{\text{odd}}(x) = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \end{cases}.$$

Use the fact that the diffusion equation is linear to split the problem over the whole line into two. Let $U(x,t) = V(x,t) + W(x,t)$, where V and W satisfy the following problems.

$$\begin{array}{ll} V_t - kV_{xx} = 0, \quad -\infty < x < \infty, \ t > 0 & W_t - kW_{xx} = f_{\text{odd}}(x,t), \quad -\infty < x < \infty, \ t > 0 \\ V(x,0) = \phi_{\text{odd}}(x) & W(x,0) = 0 \end{array}$$

The solution for V is given in section 2.4 on page 49.

$$V(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] \phi_{\text{odd}}(r)\, dr$$

According to Duhamel's principle, the solution to the inhomogeneous diffusion equation is

$$W(x,t) = \int_0^t w(x, t-s; s)\, ds,$$

where $w = w(x,t;s)$ is the solution to the associated homogeneous equation with a particular choice for the initial condition.

$$\begin{gathered} w_t - kw_{xx} = 0, \quad -\infty < x < \infty, \ t > 0 \\ w(x,0;s) = f_{\text{odd}}(x,s) \end{gathered}$$

Use the solution on page 49 again to solve for w.

$$w(x,t;s) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] f_{\text{odd}}(r,s)\, dr$$

The solution to the inhomogeneous diffusion equation is then

$$\begin{aligned} W(x,t) &= \int_0^t w(x,t-s;s)\,ds \\ &= \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4k(t-s)}\right] f_{\text{odd}}(r,s)\,dr\,ds. \end{aligned}$$

Consequently,

$$\begin{aligned} U(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right]\phi_{\text{odd}}(r)\,dr \\ + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4k(t-s)}\right] f_{\text{odd}}(r,s)\,dr\,ds. \end{aligned}$$

The solution to u is then just the restriction of U to $x>0$.

$$\begin{aligned} u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right]\phi_{\text{odd}}(r)\,dr \\ + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4k(t-s)}\right] f_{\text{odd}}(r,s)\,dr\,ds, \quad x>0 \end{aligned}$$

Our task now is to write the solution in terms of the given functions, $\phi(x)$ and $f(x,t)$. Split up each of the integrals into one over the negative values of x and one over the positive values of x and substitute the appropriate functions for ϕ_{odd} and f_{odd} in these intervals.

$$\begin{aligned} u(x,t) = \frac{1}{\sqrt{4\pi kt}}\left\{\int_{-\infty}^{0} \exp\left[-\frac{(x-r)^2}{4kt}\right][-\phi(-r)]\,dr + \int_0^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right][\phi(r)]\,dr\right\} \\ + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}}\left\{\int_{-\infty}^{0} \exp\left[-\frac{(x-r)^2}{4k(t-s)}\right][-f(-r,s)]\,dr \right. \\ \left. + \int_0^{\infty} \exp\left[-\frac{(x-r)^2}{4k(t-s)}\right][f(r,s)]\,dr\right\} ds, \quad x>0 \end{aligned}$$

Substitute $q=-r$ in the integrals from $-\infty$ to 0 and substitute $q=r$ in the integrals from 0 to ∞.

$$\begin{aligned} u(x,t) = \frac{1}{\sqrt{4\pi kt}}\left\{\int_{\infty}^{0} \exp\left[-\frac{(x+q)^2}{4kt}\right]\phi(q)\,dq + \int_0^{\infty} \exp\left[-\frac{(x-q)^2}{4kt}\right]\phi(q)\,dq\right\} \\ + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}}\left\{\int_{\infty}^{0} \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right] f(q,s)\,dq \right. \\ \left. + \int_0^{\infty} \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] f(q,s)\,dq\right\} ds, \quad x>0 \end{aligned}$$

Place minus signs appropriately to make all integrals go from 0 to ∞.

$$\begin{aligned} u(x,t) = \frac{1}{\sqrt{4\pi kt}}\left\{-\int_0^{\infty} \exp\left[-\frac{(x+q)^2}{4kt}\right]\phi(q)\,dq + \int_0^{\infty} \exp\left[-\frac{(x-q)^2}{4kt}\right]\phi(q)\,dq\right\} \\ + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}}\left\{-\int_0^{\infty} \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right] f(q,s)\,dq \right. \\ \left. + \int_0^{\infty} \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] f(q,s)\,dq\right\} ds, \quad x>0 \end{aligned}$$

Therefore, combining the integrals,

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right] \right\} \phi(q)\, dq \\ &\quad + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right] \right\} f(q,s)\, dq\, ds, \quad x > 0. \end{aligned}$$

This result can be written compactly as

$$\begin{aligned} u(x,t) &= \int_0^\infty [G(x-q,t) - G(x+q,t)]\, \phi(q)\, dq \\ &\qquad + \int_0^t \int_0^\infty [G(x-q,t-s) - G(x+q,t-s)]\, f(q,s)\, dq\, ds, \quad x > 0, \end{aligned}$$

where $G = G(x,t)$ is the Green's function for the one-dimensional diffusion equation.

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right)$$