Exercise 2

Solve the completely inhomogeneous diffusion problem on the half-line

$$\begin{aligned} v_t - kv_{xx} &= f(x,t) \qquad \text{for } 0 < x < \infty, \quad 0 < t < \infty \\ v(0,t) &= h(t) \qquad v(x,0) = \phi(x), \end{aligned}$$

by carrying out the subtraction method begun in the text.

Solution

Following the subtraction method in the text, make the substitution,

$$V(x,t) = v(x,t) - h(t),$$

in order to make the Dirichlet boundary condition homogeneous. This will allow us to use the method of reflection to solve the problem as in the previous exercise. Find the derivatives of v in terms of this new variable.

$$\begin{aligned} V_t &= v_t - h'(t) \quad \rightarrow \quad v_t = V_t + h'(t) \\ V_x &= v_x \\ V_{xx} &= v_{xx} \end{aligned}$$

The PDE V satisfies is then

$$[V_t + h'(t)] - kV_{xx} = f(x,t),$$

or

$$V_t - kV_{xx} = f(x,t) - h'(t), \quad 0 < x < \infty, \ t > 0.$$

The initial and boundary conditions associated with this PDE are

$$\begin{aligned} V(x,0) &= v(x,0) - h(0) = \phi(x) - h(0) \\ V(0,t) &= v(0,t) - h(t) = h(t) - h(t) = 0. \end{aligned}$$

Solution by the Method of Reflection

The strategy here is to solve this same problem over the whole line ($-\infty < x < \infty$) using the odd extensions of the functions, $f(x,t) - h'(t)$ and $\phi(x) - h(0)$, and then to restrict the domain to $x > 0$ to get the solution to V. Once we have $V(x,t)$, we can add $h(t)$ to it to get $v(x,t)$, the desired solution. The point of using the odd extensions of the functions is so that the Dirichlet boundary condition $V(0,t) = 0$ is satisfied. Let $W(x,t)$ be the solution to the same problem over the whole line,

$$\begin{aligned} W_t - kW_{xx} &= F_{\text{odd}}(x,t), \quad -\infty < x < \infty, \ t > 0 \\ W(x,0) &= \Phi_{\text{odd}}(x), \end{aligned}$$

where

$$F_{\text{odd}}(x,t) = \begin{cases} f(x,t) - h'(t) & x > 0 \\ -f(-x,t) + h'(t) & x < 0 \end{cases} \quad \text{and} \quad \Phi_{\text{odd}}(x) = \begin{cases} \phi(x) - h(0) & x > 0 \\ -\phi(-x) + h(0) & x < 0 \end{cases}.$$

Use the fact that the diffusion equation is linear to split the problem over the whole line into two. Let $W(x,t) = w(x,t) + u(x,t)$, where w and u satisfy the following problems.

$$\begin{aligned} & w_t - kw_{xx} = 0, \quad -\infty < x < \infty, \ t > 0 \qquad && u_t - ku_{xx} = F_{\text{odd}}(x,t), \quad -\infty < x < \infty, \ t > 0 \\ & w(x,0) = \Phi_{\text{odd}}(x) && u(x,0) = 0 \end{aligned}$$

The solution for w is given in section 2.4 on page 49.

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] \Phi_{\text{odd}}(r)\, dr$$

According to Duhamel's principle, the solution to the inhomogeneous diffusion equation is

$$u(x,t) = \int_0^t U(x, t-s; s)\, ds,$$

where $U = U(x,t;s)$ is the solution to the associated homogeneous equation with a particular choice for the initial condition.

$$\begin{gathered} U_t - kU_{xx} = 0, \quad -\infty < x < \infty, \ t > 0 \\ U(x,0;s) = F_{\text{odd}}(x,s) \end{gathered}$$

Use the solution on page 49 again to solve for U.

$$U(x,t;s) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] F_{\text{odd}}(r,s)\, dr$$

The solution to the inhomogeneous diffusion equation is then

$$\begin{aligned} u(x,t) &= \int_0^t U(x,t-s;s)\, ds \\ &= \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4k(t-s)}\right] F_{\text{odd}}(r,s)\, dr\, ds. \end{aligned}$$

Consequently,

$$\begin{aligned} W(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] \Phi_{\text{odd}}(r)\, dr \\ + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4k(t-s)}\right] F_{\text{odd}}(r,s)\, dr\, ds. \end{aligned}$$

The solution to V is then just the restriction of W to $x > 0$.

$$\begin{aligned} V(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4kt}\right] \Phi_{\text{odd}}(r)\, dr \\ + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-r)^2}{4k(t-s)}\right] F_{\text{odd}}(r,s)\, dr\, ds, \quad x > 0 \end{aligned}$$

Our task now is to write the solution in terms of the given functions, $\phi(x)$ and $f(x,t)$. Split up each of the integrals into one over the negative values of x and one over the positive values of x and substitute the appropriate functions for Φ_{odd} and F_{odd} in these intervals.

$$\begin{aligned}V(x,t) = \frac{1}{\sqrt{4\pi kt}}&\left\{\int_{-\infty}^{0}\exp\left[-\frac{(x-r)^2}{4kt}\right][-\phi(-r)+h(0)]\,dr+\int_{0}^{\infty}\exp\left[-\frac{(x-r)^2}{4kt}\right][\phi(r)-h(0)]\,dr\right\}\\&+\int_0^t\frac{1}{\sqrt{4\pi k(t-s)}}\left\{\int_{-\infty}^{0}\exp\left[-\frac{(x-r)^2}{4k(t-s)}\right][-f(-r,s)+h'(s)]\,dr\right.\\&\qquad\left.+\int_0^\infty\exp\left[-\frac{(x-r)^2}{4k(t-s)}\right][f(r,s)-h'(s)]\,dr\right\}ds,\quad x>0\end{aligned}$$

Substitute $q=-r$ in the integrals from $-\infty$ to 0 and substitute $q=r$ in the integrals from 0 to ∞.

$$\begin{aligned}V(x,t) = \frac{1}{\sqrt{4\pi kt}}&\left\{\int_{\infty}^{0}\exp\left[-\frac{(x+q)^2}{4kt}\right][-\phi(q)+h(0)](-dq)+\int_{0}^{\infty}\exp\left[-\frac{(x-q)^2}{4kt}\right][\phi(q)-h(0)]\,dq\right\}\\&+\int_0^t\frac{1}{\sqrt{4\pi k(t-s)}}\left\{\int_{\infty}^{0}\exp\left[-\frac{(x+q)^2}{4k(t-s)}\right][-f(q,s)+h'(s)](-dq)\right.\\&\qquad\left.+\int_0^\infty\exp\left[-\frac{(x-q)^2}{4k(t-s)}\right][f(q,s)-h'(s)]\,dq\right\}ds,\quad x>0\end{aligned}$$

Use the minus signs in front of dq to make the integrals go from 0 to ∞.

$$\begin{aligned}V(x,t) = \frac{1}{\sqrt{4\pi kt}}&\left\{\int_{0}^{\infty}\exp\left[-\frac{(x+q)^2}{4kt}\right][-\phi(q)+h(0)]\,dq+\int_{0}^{\infty}\exp\left[-\frac{(x-q)^2}{4kt}\right][\phi(q)-h(0)]\,dq\right\}\\&+\int_0^t\frac{1}{\sqrt{4\pi k(t-s)}}\left\{\int_{0}^{\infty}\exp\left[-\frac{(x+q)^2}{4k(t-s)}\right][-f(q,s)+h'(s)]\,dq\right.\\&\qquad\left.+\int_0^\infty\exp\left[-\frac{(x-q)^2}{4k(t-s)}\right][f(q,s)-h'(s)]\,dq\right\}ds,\quad x>0\end{aligned}$$

Combine the integrals like so.

$$\begin{aligned}V(x,t) = \frac{1}{\sqrt{4\pi kt}}&\int_0^\infty\left\{\exp\left[-\frac{(x-q)^2}{4kt}\right]-\exp\left[-\frac{(x+q)^2}{4kt}\right]\right\}\phi(q)\,dq\\&-\frac{1}{\sqrt{4\pi kt}}\int_0^\infty\left\{\exp\left[-\frac{(x-q)^2}{4kt}\right]-\exp\left[-\frac{(x+q)^2}{4kt}\right]\right\}h(0)\,dq\\&+\int_0^t\frac{1}{\sqrt{4\pi k(t-s)}}\left\{\int_0^\infty\left\{\exp\left[-\frac{(x-q)^2}{4k(t-s)}\right]-\exp\left[-\frac{(x+q)^2}{4k(t-s)}\right]\right\}f(q,s)\,dq\right.\\&\qquad\left.-\int_0^\infty\left\{\exp\left[-\frac{(x-q)^2}{4k(t-s)}\right]-\exp\left[-\frac{(x+q)^2}{4k(t-s)}\right]\right\}h'(s)\,dq\right\}ds,\quad x>0\end{aligned}$$

Distribute the integral in ds.

$$\begin{aligned} V(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty & \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right] \right\} \phi(q)\, dq \\ & - \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right] \right\} h(0)\, dq \\ & + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right] \right\} f(q,s)\, dq\, ds \\ & - \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right] \right\} h'(s)\, dq\, ds, \quad x > 0 \end{aligned}$$

Switch the order of integration in the last integral.

$$\begin{aligned} V(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty & \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right] \right\} \phi(q)\, dq \\ & - \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right] \right\} h(0)\, dq \\ & + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right] \right\} f(q,s)\, dq\, ds \\ & - \int_0^\infty \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right] \right\} h'(s)\, ds\, dq, \quad x > 0 \end{aligned}$$

Integrate the last integral by parts in order to remove the derivative from h.

$$\begin{aligned} V(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty & \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right] \right\} \phi(q)\, dq \\ & - \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right] \right\} h(0)\, dq \\ & + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right] \right\} f(q,s)\, dq\, ds \\ & - \int_0^\infty \frac{1}{\sqrt{4\pi k(t-s)}} \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right] \right\} h(s)\bigg|_0^t\, dq \\ & + \int_0^\infty \int_0^t \frac{\partial}{\partial s} \left\{ \frac{1}{\sqrt{4\pi k(t-s)}} \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right] \right\} \right\} h(s)\, ds\, dq, \\ & x > 0 \end{aligned}$$

The terms with $h(0)$ cancel.

$$\begin{aligned}
V(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty & \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right]\right\} \phi(q)\,dq \\
& - \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right]\right\} h(0)\,dq \\
& + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right]\right\} f(q,s)\,dq\,ds \\
& - \int_0^\infty \frac{1}{\sqrt{4\pi k(t-s)}} \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right]\right\} h(s)\bigg|_{s=t}\,dq \\
& + \int_0^\infty \frac{1}{\sqrt{4\pi kt}} \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right]\right\} h(0)\,dq \\
& + \int_0^\infty \int_0^t \frac{\partial}{\partial s}\left\{ \frac{1}{\sqrt{4\pi k(t-s)}} \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right]\right\}\right\} h(s)\,ds\,dq, \\
& \qquad x > 0
\end{aligned}$$

The integrand evaluated at $s = t$ is zero because the exponents of e are $-\infty$. What remains then is

$$\begin{aligned}
V(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right]\right\} \phi(q)\,dq \\
&\quad + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right]\right\} f(q,s)\,dq\,ds \\
&\quad + \int_0^\infty \int_0^t \frac{\partial}{\partial s}\left\{ \frac{1}{\sqrt{4\pi k(t-s)}} \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right]\right\}\right\} h(s)\,ds\,dq, \quad x > 0 \\
&= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right]\right\} \phi(q)\,dq \\
&\quad + \int_0^t \int_0^\infty \frac{1}{\sqrt{4\pi k(t-s)}} \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right]\right\} f(q,s)\,dq\,ds \\
&\quad + \int_0^t \int_0^\infty \frac{\partial}{\partial s}\left\{ \frac{1}{\sqrt{4\pi k(t-s)}} \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right]\right\}\right\} h(s)\,dq\,ds, \quad x > 0.
\end{aligned}$$

Therefore, since $v(x,t) = V(x,t) + h(t)$,

$$\begin{aligned}
v(x,t) = h(t) &+ \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left\{ \exp\left[-\frac{(x-q)^2}{4kt}\right] - \exp\left[-\frac{(x+q)^2}{4kt}\right]\right\} \phi(q)\,dq \\
&+ \int_0^t \int_0^\infty \frac{1}{\sqrt{4\pi k(t-s)}} \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right]\right\} f(q,s)\,dq\,ds \\
&+ \int_0^t \int_0^\infty \frac{\partial}{\partial s}\left\{ \frac{1}{\sqrt{4\pi k(t-s)}} \left\{ \exp\left[-\frac{(x-q)^2}{4k(t-s)}\right] - \exp\left[-\frac{(x+q)^2}{4k(t-s)}\right]\right\}\right\} h(s)\,dq\,ds, \quad x > 0.
\end{aligned}$$

The solution can be written compactly as

$$\begin{aligned} v(x,t) = h(t) &+ \int_0^\infty [G(x-q,t) - G(x+q,t)]\phi(q)\,dq \\ &+ \int_0^t \int_0^\infty [G(x-q,t-s) - G(x+q,t-s)]f(q,s)\,dq\,ds \\ &+ \int_0^t \int_0^\infty \frac{\partial}{\partial s}[G(x-q,t-s) - G(x+q,t-s)]h(s)\,dq\,ds, \quad x > 0, \end{aligned}$$

where $G = G(x,t)$ is the Green's function for the one-dimensional diffusion equation.

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right)$$