Exercise 1

Solve $u_{tt} = c^2 u_{xx} + xt, \quad u(x,0) = 0, \quad u_t(x,0) = 0.$

Solution

Solution by Operator Factorization

Bring $c^2 u_{xx}$ to the other side.

$$u_{tt} - c^2 u_{xx} = xt$$

Write the left side as an operator acting on u.

$$(\partial_t^2 - c^2 \partial_x^2)u = xt$$

The operator is a difference of squares, so it can be factored.

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = xt$$

Let

$$v = (\partial_t - c\partial_x)u$$

so that the PDE becomes

$$(\partial_t + c\partial_x)v = xt.$$

The second-order PDE we started with has thus been reduced to the following system of first-order PDEs that can be solved with the method of characteristics.

$$u_t - cu_x = v \tag{1}$$
$$v_t + cv_x = xt \tag{2}$$

We will solve the second one for v first, and once that is known, the first equation for u will be solved. For a function of two variables $\phi = \phi(x,t)$, its differential is defined as

$$d\phi = \frac{\partial \phi}{\partial t}\, dt + \frac{\partial \phi}{\partial x}\, dx.$$

If we divide both sides by dt, then we get the relationship between the ordinary derivative of ϕ and its partial derivatives.

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{dx}{dt} \tag{3}$$

Comparing this with equation (2), we see that along the curves in the xt-plane that satisfy

$$\frac{dx}{dt} = c, \tag{4}$$

the PDE for $v(x,t)$ reduces to an ODE.

$$\frac{dv}{dt} = xt \tag{5}$$

Because c is a constant, equation (4) can be solved by integrating both sides with respect to t.

$$x = ct + \xi, \tag{6}$$

where ξ is a characteristic coordinate. Substitute this expression for x into equation (5) to obtain an ODE that only involves t (ξ is regarded as a constant).

$$\frac{dv}{dt} = (ct + \xi)t$$

Distribute t.

$$\frac{dv}{dt} = ct^2 + \xi t$$

Integrate both sides with respect to t.

$$v(\xi, t) = \frac{ct^3}{3} + \frac{\xi t^2}{2} + f(\xi),$$

where f is an arbitrary function of the characteristic coordinate ξ. In order to write v in terms of x and t, solve equation (6) for ξ.

$$x = ct + \xi \quad \to \quad \xi = x - ct$$

Hence,

$$\begin{aligned} v(x,t) &= \frac{ct^3}{3} + \frac{t^2}{2}(x - ct) + f(x - ct) \\ &= \frac{xt^2}{2} - \frac{ct^3}{6} + f(x - ct). \end{aligned}$$

As a result, equation (1) becomes

$$u_t - cu_x = \frac{xt^2}{2} - \frac{ct^3}{6} + f(x - ct).$$

Comparing this equation with equation (3), we see that along the curves in the xt-plane that satisfy

$$\frac{dx}{dt} = -c, \tag{7}$$

the PDE for $u(x,t)$ reduces to an ODE.

$$\frac{du}{dt} = \frac{xt^2}{2} - \frac{ct^3}{6} + f(x - ct) \tag{8}$$

Because c is a constant, equation (7) can be solved by integrating both sides with respect to t.

$$x = -ct + \eta, \tag{9}$$

where η is another characteristic coordinate. Substitute this expression for x into equation (8) to obtain an ODE that only involves t (η is regarded as a constant).

$$\frac{du}{dt} = \frac{(-ct + \eta)t^2}{2} - \frac{ct^3}{6} + f(-ct + \eta - ct)$$

$$\frac{du}{dt} = \frac{\eta t^2}{2} - \frac{2ct^3}{3} + f(\eta - 2ct)$$

Integrate both sides with respect to t.

$$u(\eta, t) = \frac{\eta t^3}{6} - \frac{ct^4}{6} + \int^t f(\eta - 2cs)\, ds + g(\eta),$$

where g is an arbitrary function of the characteristic coordinate η. The integral of an arbitrary function is another arbitrary function.

$$u(\eta, t) = \frac{\eta t^3}{6} - \frac{ct^4}{6} + F(\eta - 2ct) + g(\eta),$$

In order to write u in terms of x and t, solve equation (9) for η.

$$x = -ct + \eta \quad \rightarrow \quad \eta = x + ct$$

Hence,

$$\begin{aligned} u(x, t) &= \frac{(x + ct)t^3}{6} - \frac{ct^4}{6} + F(x + ct - 2ct) + g(x + ct) \\ &= \frac{xt^3}{6} + F(x - ct) + g(x + ct). \end{aligned}$$

This is the general solution to $u_{tt} = c^2 u_{xx} + xt$. If we apply the two initial conditions, we can determine F and g. Before doing so, take a derivative of the solution with respect to t.

$$u_t(x, t) = \frac{xt^2}{2} - cF'(x - ct) + cg'(x + ct)$$

From the initial conditions we obtain the following system of equations.

$$\begin{aligned} u(x, 0) &= F(x) + g(x) = 0 \\ u_t(x, 0) &= -cF'(x) + cg'(x) = 0 \end{aligned}$$

Even though this system is in terms of x, it's really in terms of w, where w is any expression we choose.

$$\begin{aligned} F(w) + g(w) &= 0 \\ -cF'(w) + cg'(w) &= 0 \end{aligned}$$

Differentiating both sides of the first equation with respect to w, we have $F'(w) + g'(w) = 0$ or $g'(w) = -F'(w)$, so the second equation becomes

$$-cF'(w) - cF'(w) = 0 \quad \rightarrow \quad -2cF'(w) = 0 \quad \rightarrow \quad F'(w) = 0.$$

We conclude that $F(w) = C_1$ and $g(w) = -C_1$, where C_1 is a constant; consequently, $F(x - ct) = C_1$ and $g(x + ct) = -C_1$. The general solution for $u(x, t)$ becomes

$$u(x, t) = \frac{xt^3}{6} + C_1 - C_1.$$

Therefore,

$$u(x, t) = \frac{xt^3}{6}.$$

Solution by the Method of Characteristics

Bring c^2u_{xx} to the left side of the PDE.

$$u_{tt} - c^2u_{xx} = xt$$

Comparing this with the general form of a second-order PDE,

$$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G,$$

we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = xt$. The characteristic equations for a second-order PDE are given by

$$\frac{dx}{dt} = \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC}),$$

the solutions of which are known as the characteristics. Since $B^2 - 4AC = 4c^2 > 0$, the PDE is hyperbolic, so the solutions to these equations are two real and distinct families of characteristic curves in the xt-plane.

$$\begin{gathered}\frac{dx}{dt} = \frac{1}{2}(\pm\sqrt{4c^2}) \\ \frac{dx}{dt} = \frac{1}{2}(\pm 2c) \\ \frac{dx}{dt} = c \quad \text{or} \quad \frac{dx}{dt} = -c\end{gathered}$$

Integrate both sides of each equation with respect to t.

$$x = ct + C_2 \quad \text{or} \quad x = -ct + C_3$$

Now make the substitutions,

$$\begin{aligned}\xi &= x - ct = C_2 \\ \eta &= x + ct = C_3,\end{aligned}$$

so that the PDE takes the simplest form. The aim is to write u_{tt}, u_{xx}, and xt in terms of the new variables, ξ and η. Solving these two equations for x and t with elimination gives

$$\begin{aligned}x &= \frac{1}{2}(\eta + \xi) \\ t &= \frac{1}{2c}(\eta - \xi).\end{aligned}$$

Use the chain rule to write the old derivatives in terms of the new variables.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = u_\xi(-c) + u_\eta(c) = c(u_\eta - u_\xi) \\ \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi(1) + u_\eta(1) = u_\xi + u_\eta\end{aligned}$$

Find the second derivatives by using the chain rule again.

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c\frac{\partial}{\partial t}(u_\eta - u_\xi) = c\left(\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\right)(u_\eta - u_\xi) = c\left[\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi}(u_\eta - u_\xi) + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}(u_\eta - u_\xi)\right] \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(u_\xi + u_\eta) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)(u_\xi + u_\eta) = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}(u_\xi + u_\eta) + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}(u_\xi + u_\eta)\end{aligned}$$

Hence,

$$\frac{\partial^2 u}{\partial t^2} = c[(-c)(u_{\xi\eta} - u_{\xi\xi}) + (c)(u_{\eta\eta} - u_{\xi\eta})] = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$
$$\frac{\partial^2 u}{\partial x^2} = (1)(u_{\xi\xi} + u_{\xi\eta}) + (1)(u_{\xi\eta} + u_{\eta\eta}) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Substituting these expressions into the PDE, $u_{tt} - c^2 u_{xx} = xt$, we obtain

$$c^2(\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}}) - c^2(\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}}) = \left[\frac{1}{2}(\eta + \xi)\right]\left[\frac{1}{2c}(\eta - \xi)\right].$$

Simplify both sides.

$$-4c^2 u_{\xi\eta} = \frac{1}{4c}(\eta^2 - \xi^2)$$

Divide both sides by $-4c^2$.

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = -\frac{1}{16c^3}(\eta^2 - \xi^2)$$

This is known as the first canonical form of the PDE. Integrate both sides of it partially with respect to η.

$$\int^{\eta} \left.\frac{\partial^2 u}{\partial\xi\partial\eta}\right|_{\eta=s} ds = \int^{\eta} -\frac{1}{16c^3}(s^2 - \xi^2)\, ds + f(\xi),$$

where f is an arbitrary function of ξ.

$$\frac{\partial u}{\partial\xi} = -\frac{1}{16c^3}\left.\left(\frac{s^3}{3} - \xi^2 s\right)\right|^{\eta} + f(\xi)$$
$$\frac{\partial u}{\partial\xi} = -\frac{1}{16c^3}\left(\frac{\eta^3}{3} - \xi^2\eta\right) + f(\xi)$$

Now integrate both sides partially with respect to ξ.

$$\int^{\xi} \left.\frac{\partial u}{\partial\xi}\right|_{\xi=s} ds = \int^{\xi}\left[-\frac{1}{16c^3}\left(\frac{\eta^3}{3} - s^2\eta\right) + f(s)\right] ds + g(\eta),$$

where g is an arbitrary function of η.

$$u(\xi, \eta) = \left.\left[-\frac{1}{16c^3}\left(\frac{\eta^3}{3}s - \frac{s^3}{3}\eta\right) + F(s)\right]\right|^{\xi} + g(\eta)$$
$$u(\xi, \eta) = -\frac{1}{16c^3}\left(\frac{\eta^3}{3}\xi - \frac{\xi^3}{3}\eta\right) + F(\xi) + g(\eta)$$

Since u has been solved for, change back to the original variables, x and t, by substituting the expressions for ξ and η.

$$u(x, t) = -\frac{1}{16c^3}\left[\frac{(x + ct)^3}{3}(x - ct) - \frac{(x - ct)^3}{3}(x + ct)\right] + F(x - ct) + g(x + ct)$$

Expand the terms in the square brackets.

$$\begin{aligned} u(x,t) &= -\frac{1}{16c^3}\left(\frac{4}{3}ctx^3 - \frac{4}{3}c^3t^3x\right) + F(x-ct) + g(x+ct) \\ &= \frac{xt}{12}\left(t^2 - \frac{x^2}{c^2}\right) + F(x-ct) + g(x+ct) \end{aligned}$$

This is the general solution to $u_{tt} - c^2u_{xx} = xt$. We can use the two provided initial conditions to determine F and g. Before doing so, take the derivative of u with respect to t.

$$u_t(x,t) = -\frac{1}{16c^3}\left(\frac{4}{3}cx^3 - 4c^3t^2x\right) + F(x-ct) + g(x+ct)$$

From the initial conditions we obtain the following system of equations.

$$\begin{aligned} u(x,0) &= F(x) + g(x) = 0 \\ u_t(x,0) &= -\frac{x^3}{12c^2} - cF'(x) + cg'(x) = 0 \end{aligned}$$

Even though this system is in terms of x, it's really in terms of w, where w is any expression we choose.

$$\begin{aligned} F(w) + g(w) &= 0 \\ -\frac{w^3}{12c^2} - cF'(w) + cg'(w) &= 0 \end{aligned}$$

Differentiating both sides of the first equation with respect to w, we get $F'(w) + g'(w) = 0$ or $g'(w) = -F'(w)$, so the second equation becomes

$$-\frac{w^3}{12c^2} - cF'(w) - cF'(w) = 0 \quad \rightarrow \quad F'(w) = -\frac{w^3}{24c^3},$$

so

$$F(w) = -\frac{w^4}{96c^3} + C_4 \quad \Rightarrow \quad F(x-ct) = -\frac{(x-ct)^4}{96c^3} + C_4.$$

Because of the first equation, we have

$$g(w) = \frac{w^4}{96c^3} - C_4 \quad \Rightarrow \quad g(x+ct) = \frac{(x+ct)^4}{96c^3} - C_4.$$

Substitute these results into the general solution.

$$u(x,t) = \frac{xt}{12}\left(t^2 - \frac{x^2}{c^2}\right) - \frac{(x-ct)^4}{96c^3} + C_4 + \frac{(x+ct)^4}{96c^3} - C_4$$

Expanding the right side and simplifying, we obtain the same result as before.

$$u(x,t) = \frac{xt^3}{6}$$

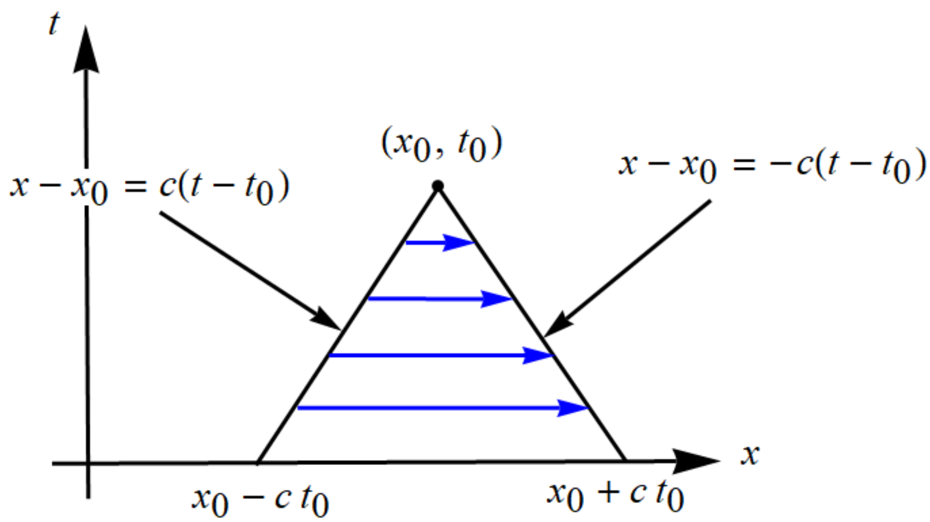
Solution by Green's Theorem

$$u_{tt} - c^2 u_{xx} = xt, \quad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = 0 \qquad u_t(x,0) = 0$$

The characteristics were found to be straight lines, $\xi = x - ct$ and $\eta = x + ct$, with slopes $\pm c$. Suppose (x_0, t_0) is the point in the xt-plane we want to evaluate u at. The equations of the lines going through this point are

$$x - x_0 = c(t - t_0)$$
$$x - x_0 = -c(t - t_0).$$

Integrate both sides of the inhomogeneous wave equation over the triangular domain D enclosed by these lines (from left to right as indicated below).



Write the double integral explicitly on the right side.

$$\iint_D (u_{tt} - c^2 u_{xx})\, dA = \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} xt\, dx\, dt$$

Rewrite the left side.

$$-\iint_D \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} xt\, dx\, dt$$

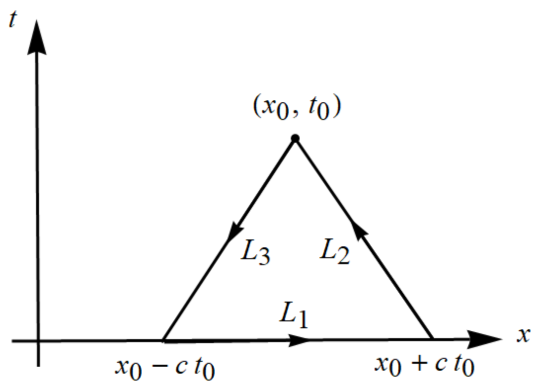
Multiply both sides by -1.

$$\iint_D \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} xt\, dx\, dt$$

Apply Green's theorem (essentially the divergence theorem in two dimensions) to the double integral on the left to turn it into a counterclockwise line integral around the triangle's boundary bdy D.

$$\oint_{\text{bdy } D} (u_t\, dx + c^2 u_x\, dt) = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} xt\, dx\, dt$$

Let L_1, L_2, and L_3 represent the legs of the triangle.



The line integral is the sum of three integrals, one over each leg.

$$\int_{L_1} (u_t\,dx + c^2 u_x\,dt) + \int_{L_2} (u_t\,dx + c^2 u_x\,dt) + \int_{L_3} (u_t\,dx + c^2 u_x\,dt) = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} xt\,dx\,dt$$

On L_1	On L_2	On L_3
$t = 0$	$x - x_0 = -c(t - t_0)$	$x - x_0 = c(t - t_0)$
$dt = 0$	$dx = -c\,dt$	$dx = c\,dt$

Replace the differentials in the integrals over L_2 and L_3.

$$\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\,dx + \int_{L_2} \left[u_t(-c\,dt) + c^2 u_x \left(-\frac{dx}{c}\right)\right] + \int_{L_3} \left[u_t(c\,dt) + c^2 u_x \left(\frac{dx}{c}\right)\right] = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} xt\,dx\,dt$$

In this exercise $u_t(x,0) = 0$, so the integral over L_1 vanishes.

$$-c\int_{L_2} \left(\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx\right) + c\int_{L_3} \left(\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx\right) = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} xt\,dx\,dt$$

The integrands are how the differential of $u = u(x,t)$ is defined.

$$-c\int_{L_2} du + c\int_{L_3} du = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} xt\,dx\,dt$$

Evaluate the integrals on the left side.

$$-c[u(x_0,t_0) - u(x_0+ct_0,0)] + c[u(x_0-ct_0,0) - u(x_0,t_0)] = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} xt\,dx\,dt$$

In this exercise $u(x,0) = 0$, so $u(x_0+ct_0,0) = 0$ and $u(x_0-ct_0,0) = 0$.

$$-2cu(x_0,t_0) = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} xt\,dx\,dt$$

Divide both sides by $-2c$.

$$u(x_0,t_0) = \frac{1}{2c}\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} xt\,dx\,dt$$

Finally, switch the roles of x and t with those of x_0 and t_0, respectively.

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x+c(t_0-t)}^{x-c(t_0-t)} x_0 t_0 \, dx_0 \, dt_0$$

Proceed to evaluate the last integral.

$$\begin{aligned}
u(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} x_0 t_0 \, dx_0 \, dt_0 \\
&= \frac{1}{2c}\int_0^t \frac{x_0^2 t_0}{2}\bigg|_{x-c(t-t_0)}^{x+c(t-t_0)} dt_0 \\
&= \frac{1}{2c}\int_0^t \frac{t_0}{2}\{[x+c(t-t_0)]^2 - [x-c(t-t_0)]^2\}\, dt_0 \\
&= \frac{1}{2c}\int_0^t \frac{t_0}{2}(4ctx - 4ct_0 x)\, dt_0 \\
&= \int_0^t t_0(tx - t_0 x)\, dt_0 \\
&= tx\int_0^t t_0\, dt_0 - x\int_0^t t_0^2\, dt_0 \\
&= tx\left(\frac{t^2}{2}\right) - x\left(\frac{t^3}{3}\right) \\
&= \frac{xt^3}{2} - \frac{xt^3}{3}
\end{aligned}$$

Therefore,

$$u(x,t) = \frac{xt^3}{6}.$$

Solution by Duhamel's Principle

$$u_{tt} - c^2 u_{xx} = xt, \quad -\infty < x < \infty, \ t > 0$$
$$u(x,0) = 0 \qquad u_t(x,0) = 0$$

According to Duhamel's principle, the solution to the inhomogeneous wave equation is

$$u(x,t) = \int_0^t U(x, t-s; s)\, ds,$$

where $U = U(x,t;s)$ is the solution to the associated homogeneous equation with a particular choice for the initial conditions.

$$U_{tt} - c^2 U_{xx} = 0, \quad -\infty < x < \infty, \ t > 0$$
$$U(x,0;s) = 0 \qquad U_t(x,0;s) = xs$$

The solution for U is given by d'Alembert's formula in section 2.1 on page 36.

$$\begin{aligned}
U(x,t;s) &= \frac{1}{2c}\int_{x-ct}^{x+ct} rs\, dr \\
&= \frac{1}{2c}\frac{r^2 s}{2}\bigg|_{x-ct}^{x+ct} \\
&= \frac{s}{4c}[(x+ct)^2 - (x-ct)^2] \\
&= \frac{s}{4c}(4ctx) \\
&= stx
\end{aligned}$$

The solution to the inhomogeneous wave equation is then

$$\begin{aligned}
u(x,t) &= \int_0^t s(t-s)x\, ds \\
&= x\int_0^t (st - s^2)\, ds \\
&= x\left(\frac{t^2}{2}t - \frac{t^3}{3}\right) \\
&= \frac{xt^3}{6}.
\end{aligned}$$

We can check that the Duhamel solution satisfies the wave equation. Use the Leibnitz rule to differentiate the integrals.

$$\begin{aligned}
u_{tt} - c^2 u_{xx} &= \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\int_0^t U(x,t-s;s)\, ds\right] - c^2\frac{\partial^2}{\partial x^2}\int_0^t U(x,t-s;s)\, ds \\
&= \frac{\partial}{\partial t}\left[\int_0^t \frac{\partial}{\partial t}U(x,t-s;s)\, ds + \underbrace{U(x,0;t)}_{=0}\cdot 1 - U(x,t;0)\cdot 0\right] - c^2\int_0^t U_{xx}(x,t-s;s)\, ds \\
&= \int_0^t \frac{\partial^2}{\partial t^2}U(x,t-s;s)\, ds + U_t(x,0;t)\cdot 1 - U_t(x,t;0)\cdot 0 - c^2\int_0^t U_{xx}(x,t-s;s)\, ds \\
&= \int_0^t [\underbrace{U_{tt}(x,t-s;s) - c^2 U_{xx}(x,t-s;s)}_{=0}]\, ds + U_t(x,0;t) = xt
\end{aligned}$$