Exercise 10

Use any method to show that $u = 1/(2c)\iint_D f$ solves the inhomogeneous wave equation on the half-line with zero initial and boundary data, where D is the domain of dependence for the half-line.

Solution

The initial boundary value problem to solve is as follows.

$$\begin{gathered} u_{tt} - c^2 u_{xx} = f(x,t), \quad 0 < x < \infty,\ t > 0 \\ u(x,0) = 0 \qquad u_t(x,0) = 0 \\ u(0,t) = 0 \end{gathered}$$

Comparing the wave equation to the general form of a second-order PDE,

$$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G,$$

we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = f(x,t)$. The characteristic equations for the second-order PDE are given by

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC}) \\ &= \frac{1}{2}(\pm\sqrt{4c^2}) \\ &= \pm c. \end{aligned}$$

Because the discriminant $B^2 - 4AC = 4c^2$ is positive, the two families of characteristic curves are real and distinct. In particular, they are lines with slopes $\pm c$ and characteristic coordinates, ξ and η, respectively.

$$\begin{aligned} \frac{dx}{dt} &= c \quad \rightarrow \quad x = ct + \xi \\ \frac{dx}{dt} &= -c \quad \rightarrow \quad x = -ct + \eta \end{aligned}$$

Suppose we are interested in evaluating u at the point (x_0, t_0). The equations of the lines going through this point are

$$\begin{aligned} x - x_0 &= c(t - t_0) \\ x - x_0 &= -c(t - t_0). \end{aligned}$$

As shown in the figure below, if (x_0, t_0) lies in the domain $x + ct > 0$, then the solution behaves as if there were no boundary. On the other hand, if (x_0, t_0) lies in the domain $x - ct < 0$, then a reflection occurs at the boundary. The solution has to be considered in each case.

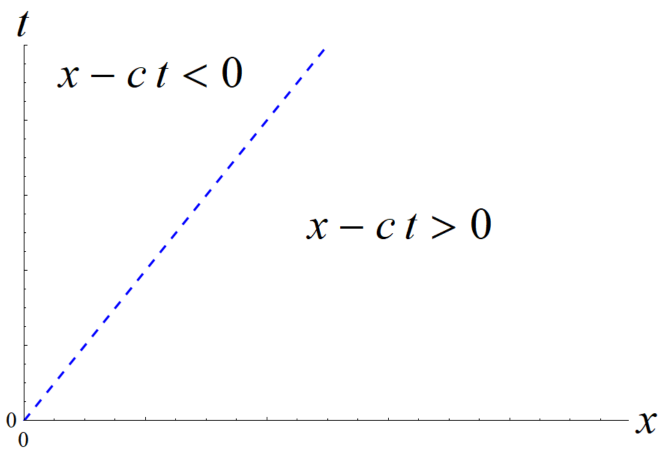
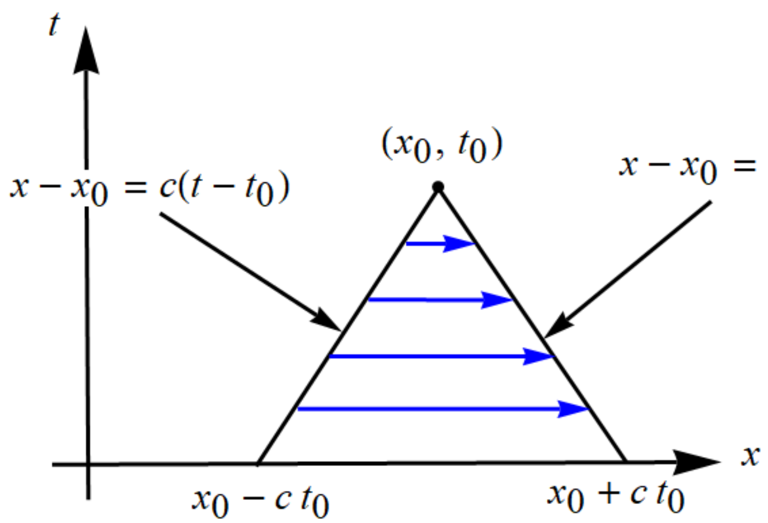


Figure 1: The presence of a boundary at $x = 0$ means we have to consider the solution to the PDE in the domains above and below the line $x - ct = 0$. The reason is that a reflection occurs for points above it but not below it.

Case 1: $x - ct > 0$

No reflection occurs in this case. Integrate both sides of the PDE over the triangular domain D_1 enclosed by the lines (from left to right as indicated below).



Write the double integral explicitly on the right side.

$$\iint\limits_{D_1} (u_{tt} - c^2 u_{xx})\, dA = \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\, dx\, dt$$

The integral will be implicit from here until the end to save space. Rewrite the left side.

$$-\iint\limits_{D_1} \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = \iint\limits_{D_1} f(x,t)\, dx\, dt$$

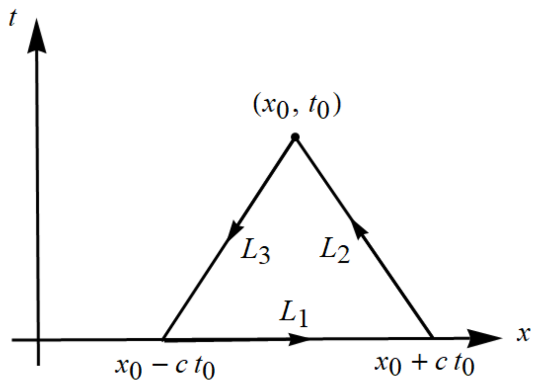
Multiply both sides by -1.

$$\iint_{D_1} \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = -\iint_{D_1} f(x,t)\, dx\, dt$$

Apply Green's theorem (essentially the divergence theorem in two dimensions) to the double integral on the left to turn it into a counterclockwise line integral around the triangle's boundary bdy D_1.

$$\oint_{\text{bdy } D_1} (u_t\, dx + c^2 u_x\, dt) = -\iint_{D_1} f(x,t)\, dx\, dt$$

Let L_1, L_2, and L_3 represent the legs of the triangle.



The line integral is the sum of three integrals, one over each leg.

$$\int_{L_1} (u_t\, dx + c^2 u_x\, dt) + \int_{L_2} (u_t\, dx + c^2 u_x\, dt) + \int_{L_3} (u_t\, dx + c^2 u_x\, dt) = -\iint_{D_1} f(x,t)\, dx\, dt$$

On L_1	On L_2	On L_3
$t = 0$	$x - x_0 = -c(t - t_0)$	$x - x_0 = c(t - t_0)$
$dt = 0$	$dx = -c\, dt$	$dx = c\, dt$

Replace the differentials in the integrals over L_2 and L_3.

$$\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\, dx + \int_{L_2} \left[u_t(-c\, dt) + c^2 u_x \left(-\frac{dx}{c}\right)\right] + \int_{L_3} \left[u_t(c\, dt) + c^2 u_x \left(\frac{dx}{c}\right)\right]$$
$$= -\iint_{D_1} f(x,t)\, dx\, dt$$

In this exercise $u_t(x,0) = 0$, so the integral over L_1 vanishes.

$$-c\int_{L_2} \left(\frac{\partial u}{\partial t}\, dt + \frac{\partial u}{\partial x}\, dx\right) + c\int_{L_3} \left(\frac{\partial u}{\partial t}\, dt + \frac{\partial u}{\partial x}\, dx\right) = -\iint_{D_1} f(x,t)\, dx\, dt$$

The integrands on the left side are how the differential of $u = u(x,t)$ is defined.

$$-c\int_{L_2} du + c\int_{L_3} du = -\iint_{D_1} f(x,t)\, dx\, dt$$

Evaluate the integrals on the left side.

$$-c[u(x_0,t_0)-u(x_0+ct_0,0)]+c[u(x_0-ct_0,0)-u(x_0,t_0)]=-\iint_{D_1} f(x,t)\,dx\,dt$$

In this exercise $u(x,0)=0$, so $u(x_0+ct_0,0)=0$ and $u(x_0-ct_0,0)=0$.

$$-2cu(x_0,t_0)=-\iint_{D_1} f(x,t)\,dx\,dt$$

Divide both sides by $-2c$ and write the double integral explicitly again.

$$u(x_0,t_0)=\frac{1}{2c}\int_0^{t_0}\int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\,dx\,dt$$

Finally, switch the roles of x and t with those of x_0 and t_0, respectively.

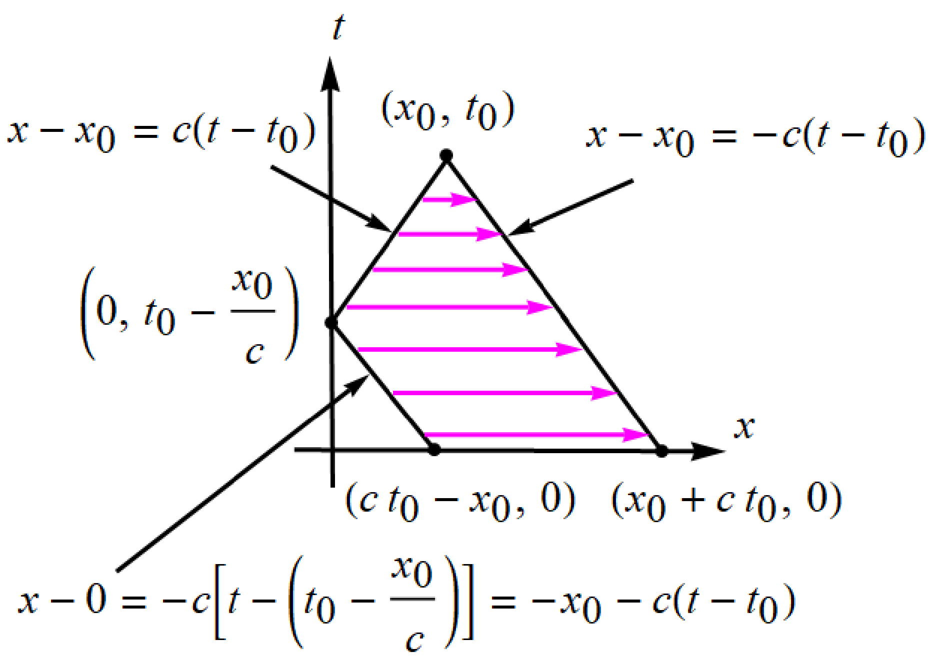
$$u(x,t)=\frac{1}{2c}\int_0^{t}\int_{x+c(t_0-t)}^{x-c(t_0-t)} f(x,t)\,dx_0\,dt_0$$

Therefore,

$$u(x,t)=\frac{1}{2c}\int_0^{t}\int_{x-c(t-t_0)}^{x+c(t-t_0)} f(x,t)\,dx_0\,dt_0,\quad x-ct>0.$$

Case 2: $x-ct<0$

Integrate both sides of the PDE over the polygonal domain D_2 enclosed by the lines (from left to right as indicated below).



Write the double integral explicitly on the right side.

$$\iint_{D_2}(u_{tt}-c^2u_{xx})\,dA=\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\,dx\,dt+\int_0^{t_0-\frac{x_0}{c}}\int_{-x_0-c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\,dx\,dt$$

The integral will be implicit from here until the end to save space. Rewrite the left side.

$$-\iint_{D_2}\left[\frac{\partial}{\partial x}(c^2u_x)-\frac{\partial}{\partial t}(u_t)\right]dA=\iint_{D_2}f(x,t)\,dx\,dt$$

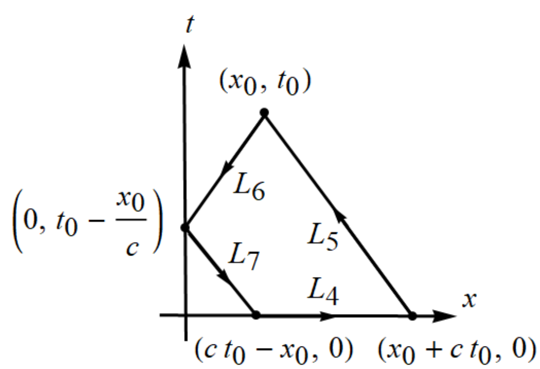
Multiply both sides by -1.

$$\iint_{D_2}\left[\frac{\partial}{\partial x}(c^2u_x)-\frac{\partial}{\partial t}(u_t)\right]dA=-\iint_{D_2}f(x,t)\,dx\,dt$$

Apply Green's theorem to the double integral on the left to turn it into a counterclockwise line integral around the polygon's boundary bdy D_2.

$$\oint_{\text{bdy }D_2}(u_t\,dx+c^2u_x\,dt)=-\iint_{D_2}f(x,t)\,dx\,dt$$

Let L_4, L_5, L_6, and L_7 represent the legs of the polygon.



The line integral is the sum of four integrals, one over each leg.

$$\int_{L_4}(u_t\,dx+c^2u_x\,dt)+\int_{L_5}(u_t\,dx+c^2u_x\,dt)+\int_{L_6}(u_t\,dx+c^2u_x\,dt)+\int_{L_7}(u_t\,dx+c^2u_x\,dt)=-\iint_{D_2}f(x,t)\,dx\,dt$$

On L_4	On L_5	On L_6	On L_7
$t=0$	$x-x_0=-c(t-t_0)$	$x-x_0=c(t-t_0)$	$x=-x_0-c(t-t_0)$
$dt=0$	$dx=-c\,dt$	$dx=c\,dt$	$dx=-c\,dt$

Replace the differentials in the integrals over L_5, L_6, and L_7.

$$\int_{ct_0-x_0}^{x_0+ct_0}u_t(x,0)\,dx+\int_{L_5}\left[u_t(-c\,dt)+c^2u_x\left(-\frac{dx}{c}\right)\right]+\int_{L_6}\left[u_t(c\,dt)+c^2u_x\left(\frac{dx}{c}\right)\right]$$
$$+\int_{L_7}\left[u_t(-c\,dt)+c^2u_x\left(-\frac{dx}{c}\right)\right]=-\iint_{D_2}f(x,t)\,dx\,dt$$

In this exercise $u_t(x,0)=0$, so the integral over L_4 vanishes.

$$-c\int_{L_5}\left(\frac{\partial u}{\partial t}dt+\frac{\partial u}{\partial x}dx\right)+c\int_{L_6}\left(\frac{\partial u}{\partial t}dt+\frac{\partial u}{\partial x}dx\right)-c\int_{L_7}\left(\frac{\partial u}{\partial t}dt+\frac{\partial u}{\partial x}dx\right)=-\iint_{D_2}f(x,t)\,dx\,dt$$

All integrands on the left side are how the differential of $u = u(x, t)$ is defined.

$$-c\int_{L_5} du + c\int_{L_6} du - c\int_{L_7} du = -\iint_{D_2} f(x,t)\,dx\,dt$$

Evaluate the integrals on the left side.

$$-c[u(x_0,t_0) - u(x_0+ct_0,0)] + c\left[u\left(0, t_0 - \frac{x_0}{c}\right) - u(x_0,t_0)\right] - c\left[u(ct_0 - x_0, 0) - u\left(0, t_0 - \frac{x_0}{c}\right)\right] = -\iint_{D_2} f(x,t)\,dx\,dt$$

In this exercise $u(x, 0) = 0$ and $u(0, t) = 0$, so $u(x_0 + ct_0, 0) = 0$ and $u(ct_0 - x_0, 0) = 0$ and $u(0, t_0 - x_0/c) = 0$.

$$-2cu(x_0,t_0) = -\iint_{D_2} f(x,t)\,dx\,dt$$

Divide both sides by $-2c$ and write out the double integral explicitly again.

$$u(x_0,t_0) = \frac{1}{2c}\left[\int_{t_0-\frac{x_0}{c}}^{t_0}\int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\,dx\,dt + \int_0^{t_0-\frac{x_0}{c}}\int_{-x_0-c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\,dx\,dt\right]$$

Finally, switch the roles of x and t with those of x_0 and t_0, respectively.

$$u(x,t) = \frac{1}{2c}\left[\int_{t-\frac{x}{c}}^{t}\int_{x+c(t_0-t)}^{x-c(t_0-t)} f(x_0,t_0)\,dx_0\,dt_0 + \int_0^{t-\frac{x}{c}}\int_{-x-c(t_0-t)}^{x-c(t_0-t)} f(x_0,t_0)\,dx_0\,dt_0\right]$$

Therefore,

$$u(x,t) = \frac{1}{2c}\left[\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-t_0)}^{x+c(t-t_0)} f(x_0,t_0)\,dx_0\,dt_0 + \int_0^{t-\frac{x}{c}}\int_{-x+c(t-t_0)}^{x+c(t-t_0)} f(x_0,t_0)\,dx_0\,dt_0\right], \quad x - ct < 0.$$

In conclusion, the solution to the initial boundary value problem is

$$u(x,t) = \begin{cases} \dfrac{1}{2c}\left[\displaystyle\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-t_0)}^{x+c(t-t_0)} f(x_0,t_0)\,dx_0\,dt_0 + \int_0^{t-\frac{x}{c}}\int_{-x+c(t-t_0)}^{x+c(t-t_0)} f(x_0,t_0)\,dx_0\,dt_0\right] & \text{if } x - ct < 0 \\ \dfrac{1}{2c}\displaystyle\int_0^{t}\int_{x-c(t-t_0)}^{x+c(t-t_0)} f(x,t)\,dx_0\,dt_0 & \text{if } x - ct > 0 \end{cases}.$$