Exercise 11

Show by direct substitution that $u(x,t) = h(t - x/c)$ for $x < ct$ and $u(x,t) = 0$ for $x \geq ct$ solves the wave equation on the half-line $(0, \infty)$ with zero initial data and boundary condition $u(0,t) = h(t)$.

Solution

Here we have to verify that

$$u(x,t) = \begin{cases} h(t - x/c) & x < ct \\ 0 & x \geq ct \end{cases}$$

solves the wave equation,

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty,$$

with the conditions,

$$u(x,0) = 0 \quad \text{and} \quad u_t(x,0) = 0 \quad \text{and} \quad u(0,t) = h(t).$$

Differentiate $u(x,t)$ with respect to t twice.

$$u_t(x,t) = \begin{cases} h'(t - x/c) & x < ct \\ 0 & x \geq ct \end{cases} \quad \rightarrow \quad u_{tt}(x,t) = \begin{cases} h''(t - x/c) & x < ct \\ 0 & x \geq ct \end{cases}$$

Differentiate $u(x,t)$ with respect to x twice.

$$u_x(x,t) = \begin{cases} h'(t - x/c) \cdot \left(-\frac{1}{c}\right) & x < ct \\ 0 & x \geq ct \end{cases} \quad \rightarrow \quad u_{xx}(x,t) = \begin{cases} h''(t - x/c) \cdot \left(-\frac{1}{c}\right)^2 & x < ct \\ 0 & x \geq ct \end{cases}$$

For $x < ct$,

$$u_{tt} = h''(t - x/c) = \cancel{c^2} h''(t - x/c) \cdot \cancel{\left(-\frac{1}{c}\right)^2} = c^2 u_{xx}.$$

For $x \geq ct$,

$$u_{tt} = 0 = c^2 u_{xx}.$$

We conclude that $u(x,t)$ satisfies the wave equation. When $t = 0$ the condition $x \geq ct$ applies, so

$$u(x,0) = 0$$
$$u_t(x,0) = 0.$$

When $x = 0$ the condition $x < ct$ applies, so

$$u(0,t) = h(t - 0/c) = h(t).$$