

Exercise 12

Derive the solution of the fully inhomogeneous wave equation on the half-line

$$\begin{aligned}v_{tt} - c^2 v_{xx} &= f(x, t) \quad \text{in } 0 < x < \infty \\v(x, 0) &= \phi(x), \quad v_t(x, 0) = \psi(x) \\v(0, t) &= h(t),\end{aligned}$$

by means of the method using Green's theorem. (*Hint:* Integrate over the domain of dependence.)

Solution

It will be assumed that $t > 0$. Comparing the wave equation to the general form of a second-order PDE,

$$Av_{tt} + Bv_{xt} + Cv_{xx} + Dv_t + Ev_x + Fv = G,$$

we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = f(x, t)$. The characteristic equations for the second-order PDE are given by

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC}) \\&= \frac{1}{2}(\pm\sqrt{4c^2}) \\&= \pm c.\end{aligned}$$

Because the discriminant $B^2 - 4AC = 4c^2$ is positive, the two families of characteristic curves are real and distinct. In particular, they are lines with slopes $\pm c$ and characteristic coordinates, ξ and η , respectively.

$$\begin{aligned}\frac{dx}{dt} = c &\rightarrow x = ct + \xi \\ \frac{dx}{dt} = -c &\rightarrow x = -ct + \eta\end{aligned}$$

Suppose we are interested in evaluating v at the point (x_0, t_0) . The equations of the lines going through this point are

$$\begin{aligned}x - x_0 &= c(t - t_0) \\x - x_0 &= -c(t - t_0).\end{aligned}$$

As shown in the figure below, if (x_0, t_0) lies in the domain $x + ct > 0$, then the solution behaves as if there were no boundary. On the other hand, if (x_0, t_0) lies in the domain $x - ct < 0$, then a reflection occurs at the boundary. The solution has to be considered in each case.

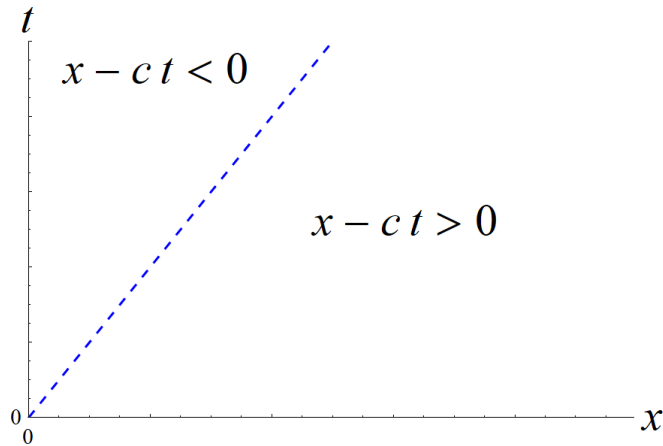
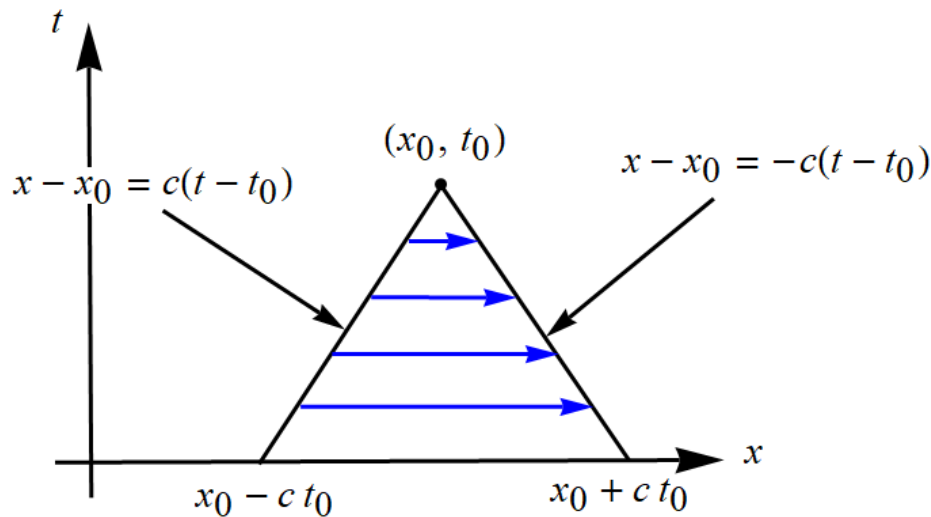


Figure 1: The presence of a boundary at $x = 0$ means we have to consider the solution to the PDE in the domains above and below the line $x - ct = 0$. The reason is that a reflection occurs for points above it but not below it.

Case 1: $x - ct > 0$

No reflection occurs in this case. Integrate both sides of the PDE over the triangular domain D_1 enclosed by the lines (from left to right as indicated below).



Write the double integral explicitly on the right side.

$$\iint_{D_1} (v_{tt} - c^2 v_{xx}) dA = \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x, t) dx dt$$

The integral will be implicit from here until the end to save space. Rewrite the left side.

$$-\iint_{D_1} \left[\frac{\partial}{\partial x} (c^2 v_x) - \frac{\partial}{\partial t} (v_t) \right] dA = \iint_{D_1} f(x, t) dx dt$$

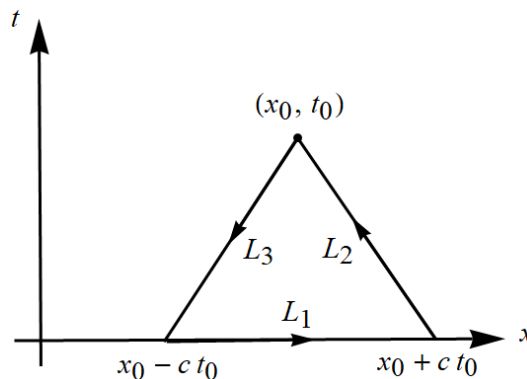
Multiply both sides by -1 .

$$\iint_{D_1} \left[\frac{\partial}{\partial x} (c^2 v_x) - \frac{\partial}{\partial t} (v_t) \right] dA = - \iint_{D_1} f(x, t) dx dt$$

Apply Green's theorem (essentially the divergence theorem in two dimensions) to the double integral on the left to turn it into a counterclockwise line integral around the triangle's boundary $\text{bdy } D_1$.

$$\oint_{\text{bdy } D_1} (v_t dx + c^2 v_x dt) = - \iint_{D_1} f(x, t) dx dt$$

Let L_1 , L_2 , and L_3 represent the legs of the triangle.



The line integral is the sum of three integrals, one over each leg.

$$\int_{L_1} (v_t dx + c^2 v_x dt) + \int_{L_2} (v_t dx + c^2 v_x dt) + \int_{L_3} (v_t dx + c^2 v_x dt) = - \iint_{D_1} f(x, t) dx dt$$

On L_1	On L_2	On L_3
$t = 0$	$x - x_0 = -c(t - t_0)$	$x - x_0 = c(t - t_0)$
$dt = 0$	$dx = -c dt$	$dx = c dt$

Replace the differentials in the integrals over L_2 and L_3 .

$$\int_{x_0-ct_0}^{x_0+ct_0} v_t(x, 0) dx + \int_{L_2} \left[v_t(-c dt) + c^2 v_x \left(-\frac{dx}{c} \right) \right] + \int_{L_3} \left[v_t(c dt) + c^2 v_x \left(\frac{dx}{c} \right) \right] = - \iint_{D_1} f(x, t) dx dt$$

In this exercise $v_t(x, 0) = \psi(x)$.

$$\int_{x_0-ct_0}^{x_0+ct_0} \psi(x) dx - c \int_{L_2} \left(\frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial x} dx \right) + c \int_{L_3} \left(\frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial x} dx \right) = - \iint_{D_1} f(x, t) dx dt$$

The second and third integrands on the left side are how the differential of $v = v(x, t)$ is defined.

$$\int_{x_0-ct_0}^{x_0+ct_0} \psi(x) dx - c \int_{L_2} dv + c \int_{L_3} dv = - \iint_{D_1} f(x, t) dx dt$$

Evaluate the second and third integrals on the left side.

$$\int_{x_0-ct_0}^{x_0+ct_0} \psi(x) dx - c[v(x_0, t_0) - v(x_0 + ct_0, 0)] + c[v(x_0 - ct_0, 0) - v(x_0, t_0)] = - \iint_{D_1} f(x, t) dx dt$$

In this exercise $v(x, 0) = \phi(x)$, so $v(x_0 + ct_0, 0) = \phi(x_0 + ct_0)$ and $v(x_0 - ct_0, 0) = \phi(x_0 - ct_0)$.

$$\int_{x_0-ct_0}^{x_0+ct_0} \psi(x) dx - 2cv(x_0, t_0) + c[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] = - \iint_{D_1} f(x, t) dx dt$$

Solve this equation for $2cv(x_0, t_0)$.

$$2cv(x_0, t_0) = c[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] + \int_{x_0-ct_0}^{x_0+ct_0} \psi(x) dx + \iint_{D_1} f(x, t) dx dt$$

Divide both sides by $2c$ and write the double integral explicitly again.

$$v(x_0, t_0) = \frac{1}{2}[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] + \frac{1}{2c} \int_{x_0-ct_0}^{x_0+ct_0} \psi(x) dx + \frac{1}{2c} \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x, t) dx dt$$

Finally, switch the roles of x and t with those of x_0 and t_0 , respectively.

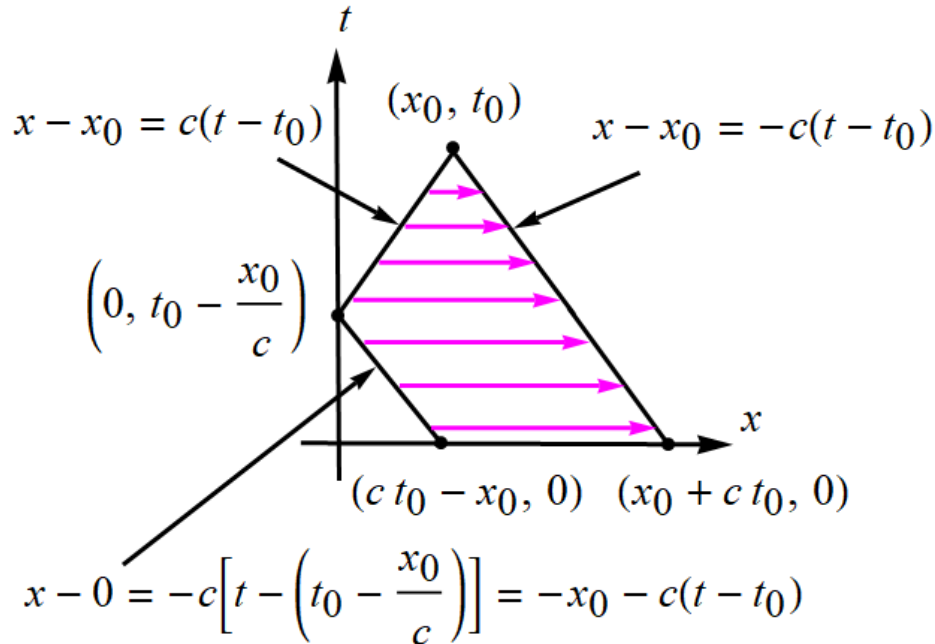
$$v(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x_0) dx_0 + \frac{1}{2c} \int_0^t \int_{x+c(t_0-t)}^{x-c(t_0-t)} f(x_0, t_0) dx_0 dt_0$$

Therefore,

$$v(x, t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x_0) dx_0 + \frac{1}{2c} \int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} f(x_0, t_0) dx_0 dt_0, \quad x-ct > 0.$$

Case 2: $x - ct < 0$

Integrate both sides of the PDE over the polygonal domain D_2 enclosed by the lines (from left to right as indicated below).



Write the double integral explicitly on the right side.

$$\iint_{D_2} (v_{tt} - c^2 v_{xx}) dA = \int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0 + c(t-t_0)}^{x_0 - c(t-t_0)} f(x, t) dx dt + \int_0^{t_0 - \frac{x_0}{c}} \int_{-x_0 - c(t-t_0)}^{x_0 - c(t-t_0)} f(x, t) dx dt$$

The integral will be implicit from here until the end to save space. Rewrite the left side.

$$- \iint_{D_2} \left[\frac{\partial}{\partial x} (c^2 v_x) - \frac{\partial}{\partial t} (v_t) \right] dA = \iint_{D_2} f(x, t) dx dt$$

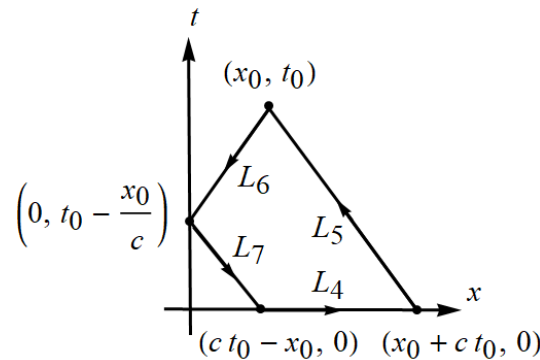
Multiply both sides by -1 .

$$\iint_{D_2} \left[\frac{\partial}{\partial x} (c^2 v_x) - \frac{\partial}{\partial t} (v_t) \right] dA = - \iint_{D_2} f(x, t) dx dt$$

Apply Green's theorem to the double integral on the left to turn it into a counterclockwise line integral around the polygon's boundary $\text{bdy } D_2$.

$$\oint_{\text{bdy } D_2} (v_t dx + c^2 v_x dt) = - \iint_{D_2} f(x, t) dx dt$$

Let $L_4, L_5, L_6,$ and L_7 represent the legs of the polygon.



The line integral is the sum of four integrals, one over each leg.

$$\int_{L_4} (v_t dx + c^2 v_x dt) + \int_{L_5} (v_t dx + c^2 v_x dt) + \int_{L_6} (v_t dx + c^2 v_x dt) + \int_{L_7} (v_t dx + c^2 v_x dt) = - \iint_{D_2} f(x, t) dx dt$$

On L_4	On L_5	On L_6	On L_7
$t = 0$	$x - x_0 = -c(t - t_0)$	$x - x_0 = c(t - t_0)$	$x = -x_0 - c(t - t_0)$
$dt = 0$	$dx = -c dt$	$dx = c dt$	$dx = -c dt$

Replace the differentials in the integrals over $L_5, L_6,$ and L_7 .

$$\int_{ct_0 - x_0}^{x_0 + ct_0} v_t(x, 0) dx + \int_{L_5} \left[v_t(-c dt) + c^2 v_x \left(-\frac{dx}{c} \right) \right] + \int_{L_6} \left[v_t(c dt) + c^2 v_x \left(\frac{dx}{c} \right) \right] + \int_{L_7} \left[v_t(-c dt) + c^2 v_x \left(-\frac{dx}{c} \right) \right] = - \iint_{D_2} f(x, t) dx dt$$

In this exercise $v_t(x, 0) = \psi(x)$.

$$\int_{ct_0 - x_0}^{x_0 + ct_0} \psi(x) dx - c \int_{L_5} \left(\frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial x} dx \right) + c \int_{L_6} \left(\frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial x} dx \right) - c \int_{L_7} \left(\frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial x} dx \right) = - \iint_{D_2} f(x, t) dx dt$$

The second, third, and fourth integrands on the left side are how the differential of $v = v(x, t)$ is defined.

$$\int_{ct_0 - x_0}^{x_0 + ct_0} \psi(x) dx - c \int_{L_5} dv + c \int_{L_6} dv - c \int_{L_7} dv = - \iint_{D_2} f(x, t) dx dt$$

Evaluate the second, third, and fourth integrals on the left side.

$$\int_{ct_0 - x_0}^{x_0 + ct_0} \psi(x) dx - c[v(x_0, t_0) - v(x_0 + ct_0, 0)] + c \left[v \left(0, t_0 - \frac{x_0}{c} \right) - v(x_0, t_0) \right] - c \left[v(ct_0 - x_0, 0) - v \left(0, t_0 - \frac{x_0}{c} \right) \right] = - \iint_{D_2} f(x, t) dx dt$$

In this exercise $v(x, 0) = \phi(x)$ and $v(0, t) = h(t)$, so $v(x_0 + ct_0, 0) = \phi(x_0 + ct_0)$ and $v(ct_0 - x_0, 0) = \phi(ct_0 - x_0)$ and $v(0, t_0 - x_0/c) = h(t_0 - x_0/c)$.

$$\int_{ct_0 - x_0}^{x_0 + ct_0} \psi(x) dx - 2cv(x_0, t_0) + 2ch\left(t_0 - \frac{x_0}{c}\right) + c[\phi(x_0 + ct_0) - \phi(ct_0 - x_0)] = - \iint_{D_2} f(x, t) dx dt$$

Solve this equation for $2cv(x_0, t_0)$.

$$2cv(x_0, t_0) = 2ch\left(t_0 - \frac{x_0}{c}\right) + c[\phi(x_0 + ct_0) - \phi(ct_0 - x_0)] + \int_{ct_0 - x_0}^{x_0 + ct_0} \psi(x) dx + \iint_{D_2} f(x, t) dx dt$$

Divide both sides by $2c$ and write out the double integral explicitly again.

$$v(x_0, t_0) = h\left(t_0 - \frac{x_0}{c}\right) + \frac{1}{2}[\phi(x_0 + ct_0) - \phi(ct_0 - x_0)] + \frac{1}{2c} \int_{ct_0 - x_0}^{x_0 + ct_0} \psi(x) dx \\ + \frac{1}{2c} \left[\int_{t_0 - \frac{x_0}{c}}^{t_0} \int_{x_0 + c(t-t_0)}^{x_0 - c(t-t_0)} f(x, t) dx dt + \int_0^{t_0 - \frac{x_0}{c}} \int_{-x_0 - c(t-t_0)}^{x_0 - c(t-t_0)} f(x, t) dx dt \right]$$

Finally, switch the roles of x and t with those of x_0 and t_0 , respectively.

$$v(x, t) = h\left(t - \frac{x}{c}\right) + \frac{1}{2}[\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(x_0) dx_0 \\ + \frac{1}{2c} \left[\int_{t - \frac{x}{c}}^t \int_{x+c(t_0-t)}^{x-c(t_0-t)} f(x_0, t_0) dx_0 dt_0 + \int_0^{t - \frac{x}{c}} \int_{-x-c(t_0-t)}^{x-c(t_0-t)} f(x_0, t_0) dx_0 dt_0 \right] \\ = h\left(t - \frac{x}{c}\right) + \frac{1}{2}[\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(x_0) dx_0 \\ + \frac{1}{2c} \left[\int_{t - \frac{x}{c}}^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} f(x_0, t_0) dx_0 dt_0 + \int_0^{t - \frac{x}{c}} \int_{-[x-c(t-t_0)]}^{x+c(t-t_0)} f(x_0, t_0) dx_0 dt_0 \right]$$

Note that as t_0 goes from 0 to $t - x/c$, the quantity $-[x - c(t - t_0)]$ is positive, and as t_0 goes from $t - x/c$ to t , the quantity $x - c(t - t_0)$ is positive. These two double integrals may be combined by using an absolute value sign.

$$v(x, t) = h\left(t - \frac{x}{c}\right) + \frac{1}{2}[\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(x_0) dx_0 \\ + \frac{1}{2c} \int_0^t \int_{|x-c(t-t_0)|}^{x+c(t-t_0)} f(x_0, t_0) dx_0 dt_0, \quad x - ct < 0$$

Therefore, the solution to the initial boundary value problem is

$$v(x, t) = \begin{cases} h\left(t - \frac{x}{c}\right) + \frac{1}{2}[\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(x_0) dx_0 \\ \quad + \frac{1}{2c} \int_0^t \int_{|x-c(t-t_0)|}^{x+c(t-t_0)} f(x_0, t_0) dx_0 dt_0 & \text{if } x - ct < 0. \\ \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(x_0) dx_0 + \frac{1}{2c} \int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} f(x_0, t_0) dx_0 dt_0 & \text{if } x - ct > 0 \end{cases}$$