Exercise 13

Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \infty$,
$u(0,t) = t^2, \quad u(x,0) = x, \quad u_t(x,0) = 0.$

Solution

It will be assumed that $t > 0$. Bring $c^2 u_{xx}$ to the left side.

$$u_{tt} - c^2 u_{xx} = 0$$

Comparing the wave equation to the general form of a second-order PDE,

$$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G,$$

we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations for a second-order PDE are given by

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC}) \\ &= \frac{1}{2}(\pm\sqrt{4c^2}) \\ &= \pm c. \end{aligned}$$

Because the discriminant $B^2 - 4AC = 4c^2$ is positive, the two families of characteristic curves are real and distinct. In particular, they are lines with slopes $\pm c$ and characteristic coordinates, ξ and η, respectively.

$$\begin{aligned} \frac{dx}{dt} &= c \quad \rightarrow \quad x = ct + \xi \\ \frac{dx}{dt} &= -c \quad \rightarrow \quad x = -ct + \eta \end{aligned}$$

Suppose we are interested in evaluating u at the point (x_0, t_0). The equations of the lines going through this point are

$$\begin{aligned} x - x_0 &= c(t - t_0) \\ x - x_0 &= -c(t - t_0). \end{aligned}$$

As shown in the figure below, if (x_0, t_0) lies in the domain $x + ct > 0$, then the solution behaves as if there were no boundary. On the other hand, if (x_0, t_0) lies in the domain $x - ct < 0$, then a reflection occurs at the boundary. The solution has to be considered in each case.

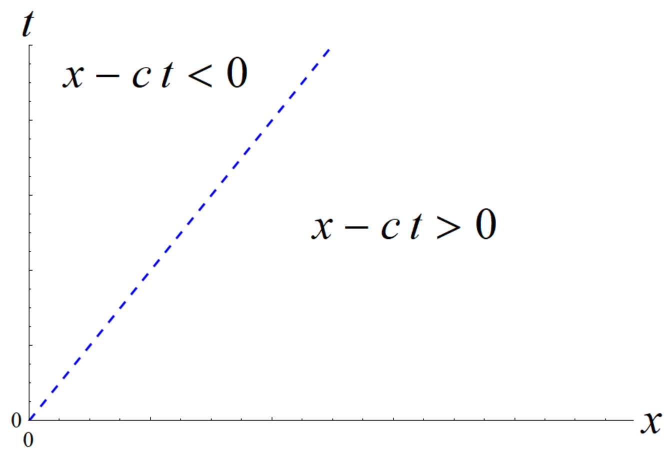
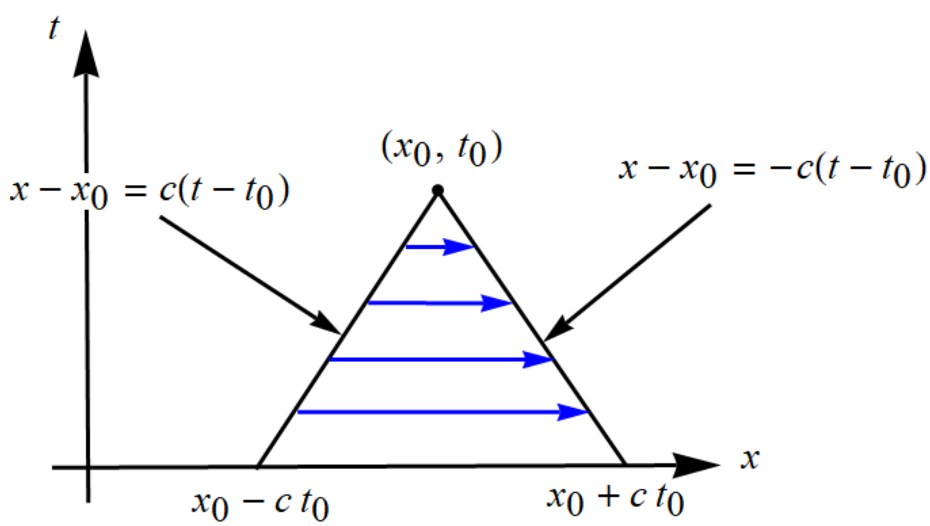


Figure 1: The presence of a boundary at $x = 0$ means we have to consider the solution to the PDE in the domains above and below the line $x - ct = 0$. The reason is that a reflection occurs for points above it but not below it.

Case 1: $x - ct > 0$



No reflection occurs in this case. Integrate both sides of the PDE over the triangular domain D_1 enclosed by the lines (from left to right as indicated above).

$$\iint_{D_1} (u_{tt} - c^2 u_{xx})\, dA = 0$$

Rewrite the left side.

$$-\iint_{D_1} \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = 0$$

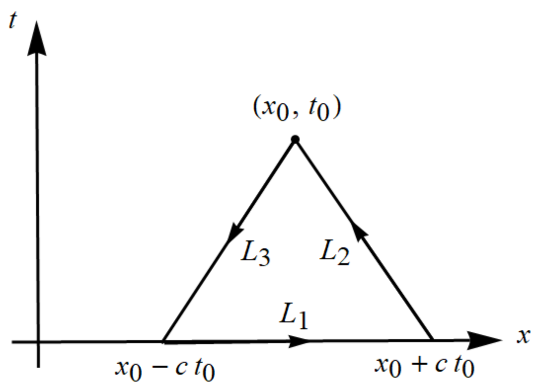
Multiply both sides by -1.

$$\iint_{D_1} \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = 0$$

Apply Green's theorem (essentially the divergence theorem in two dimensions) to the double integral to turn it into a counterclockwise line integral around the triangle's boundary bdy D_1.

$$\oint_{\text{bdy } D_1} (u_t\,dx + c^2 u_x\,dt) = 0$$

Let L_1, L_2, and L_3 represent the legs of the triangle.



The line integral is the sum of three integrals, one over each leg.

$$\int_{L_1} (u_t\,dx + c^2 u_x\,dt) + \int_{L_2} (u_t\,dx + c^2 u_x\,dt) + \int_{L_3} (u_t\,dx + c^2 u_x\,dt) = 0$$

On L_1	On L_2	On L_3
$t = 0$	$x - x_0 = -c(t - t_0)$	$x - x_0 = c(t - t_0)$
$dt = 0$	$dx = -c\,dt$	$dx = c\,dt$

Replace the differentials in the integrals over L_2 and L_3.

$$\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\,dx + \int_{L_2} \left[u_t(-c\,dt) + c^2 u_x\left(-\frac{dx}{c}\right)\right] + \int_{L_3} \left[u_t(c\,dt) + c^2 u_x\left(\frac{dx}{c}\right)\right] = 0$$

In this exercise $u_t(x,0) = 0$, so the integral over L_1 vanishes.

$$-c\int_{L_2} \left(\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx\right) + c\int_{L_3} \left(\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx\right) = 0$$

The remaining integrands are how the differential of $u = u(x,t)$ is defined.

$$-c\int_{L_2} du + c\int_{L_3} du = 0$$

Evaluate the remaining integrals.

$$-c[u(x_0,t_0) - u(x_0+ct_0,0)] + c[u(x_0-ct_0,0) - u(x_0,t_0)] = 0$$

In this exercise $u(x,0) = x$, so $u(x_0+ct_0,0) = x_0+ct_0$ and $u(x_0-ct_0,0) = x_0-ct_0$.

$$-2cu(x_0,t_0) + c[(x_0+ct_0) + (x_0-ct_0)] = 0$$

Solve this equation for $2cu(x_0, t_0)$.

$$\begin{aligned} 2cu(x_0, t_0) &= c[(x_0 + ct_0) + (x_0 - ct_0)] \\ &= 2cx_0 \end{aligned}$$

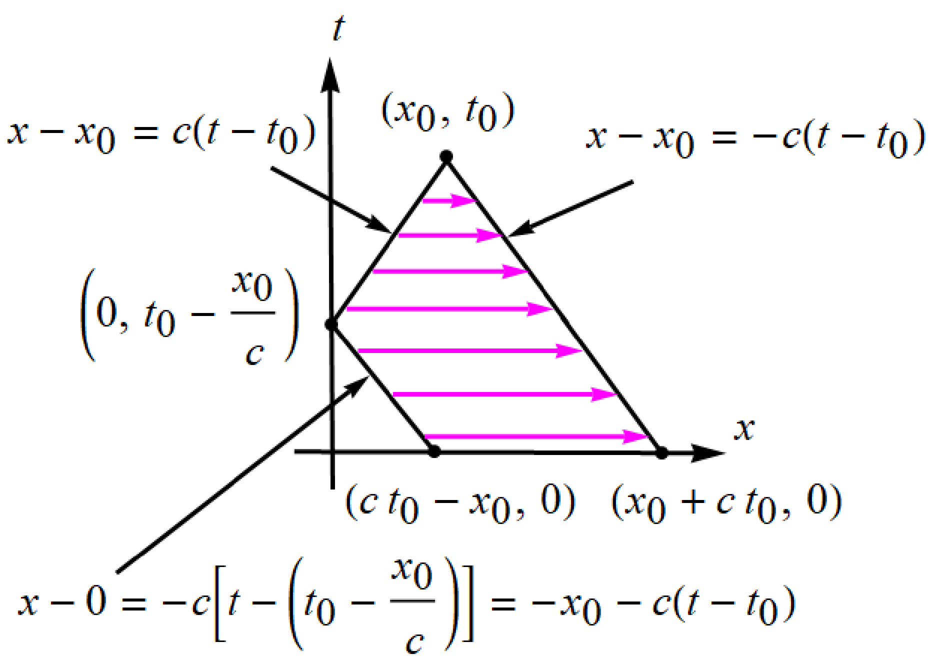
Divide both sides by $2c$.

$$u(x_0, t_0) = x_0$$

Therefore, switching the roles of x and t with those of x_0 and t_0, respectively,

$$u(x, t) = x, \quad x - ct > 0.$$

Case 2: $x - ct < 0$



Integrate both sides of the PDE over the polygonal domain D_2 enclosed by the lines (from left to right as indicated above).

$$\iint_{D_2} (u_{tt} - c^2 u_{xx})\, dA = 0$$

Rewrite the left side.

$$-\iint_{D_2} \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = 0$$

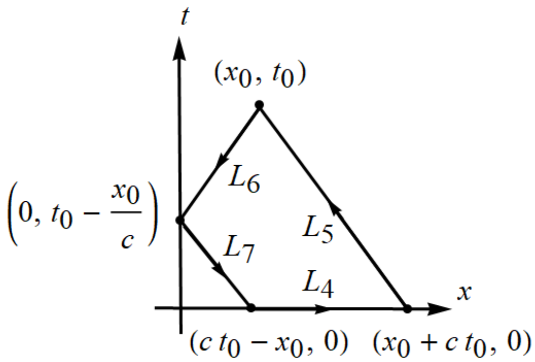
Multiply both sides by -1.

$$\iint_{D_2} \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = 0$$

Apply Green's theorem to the double integral to turn it into a counterclockwise line integral around the polygon's boundary bdy D_2.

$$\oint_{\text{bdy } D_2} (u_t\, dx + c^2 u_x\, dt) = 0$$

Let L_4, L_5, L_6, and L_7 represent the legs of the polygon.



The line integral is the sum of four integrals, one over each leg.

$$\int_{L_4}(u_t\,dx + c^2u_x\,dt) + \int_{L_5}(u_t\,dx + c^2u_x\,dt) + \int_{L_6}(u_t\,dx + c^2u_x\,dt) + \int_{L_7}(u_t\,dx + c^2u_x\,dt) = 0$$

On L_4	On L_5	On L_6	On L_7
$t = 0$	$x - x_0 = -c(t - t_0)$	$x - x_0 = c(t - t_0)$	$x = -x_0 - c(t - t_0)$
$dt = 0$	$dx = -c\,dt$	$dx = c\,dt$	$dx = -c\,dt$

Replace the differentials in the integrals over L_5, L_6, and L_7.

$$\int_{ct_0-x_0}^{x_0+ct_0} u_t(x,0)\,dx + \int_{L_5}\left[u_t(-c\,dt) + c^2u_x\left(-\frac{dx}{c}\right)\right] + \int_{L_6}\left[u_t(c\,dt) + c^2u_x\left(\frac{dx}{c}\right)\right] + \int_{L_7}\left[u_t(-c\,dt) + c^2u_x\left(-\frac{dx}{c}\right)\right] = 0$$

In this exercise $u_t(x,0) = 0$, so the integral over L_4 vanishes.

$$-c\int_{L_5}\left(\frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx\right) + c\int_{L_6}\left(\frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx\right) - c\int_{L_7}\left(\frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx\right) = 0$$

The remaining integrands on the left side are how the differential of $u = u(x,t)$ is defined.

$$-c\int_{L_5} du + c\int_{L_6} du - c\int_{L_7} du = 0$$

Evaluate the remaining integrals.

$$-c[u(x_0,t_0) - u(x_0+ct_0,0)] + c\left[u\left(0, t_0 - \frac{x_0}{c}\right) - u(x_0,t_0)\right] - c\left[u(ct_0 - x_0, 0) - u\left(0, t_0 - \frac{x_0}{c}\right)\right] = 0$$

In this exercise $u(x,0) = x$ and $u(0,t) = t^2$, so $u(x_0 + ct_0, 0) = x_0 + ct_0$ and $u(ct_0 - x_0, 0) = ct_0 - x_0$ and $u(0, t_0 - x_0/c) = (t_0 - x_0/c)^2$.

$$-2cu(x_0,t_0) + 2c\left(t_0 - \frac{x_0}{c}\right)^2 + c[(x_0 + ct_0) - (ct_0 - x_0)] = 0$$

Solve this equation for $2cu(x_0, t_0)$.

$$2cu(x_0, t_0) = 2c\left(t_0 - \frac{x_0}{c}\right)^2 + c[(x_0 + ct_0) - (ct_0 - x_0)]$$

Divide both sides by $2c$.

$$\begin{aligned} u(x_0, t_0) &= \left(t_0 - \frac{x_0}{c}\right)^2 + \frac{1}{2}[(x_0 + ct_0) - (ct_0 - x_0)] \\ &= \left(t_0 - \frac{x_0}{c}\right)^2 + \frac{1}{2}(2x_0) \\ &= x_0 + \left(t_0 - \frac{x_0}{c}\right)^2 \end{aligned}$$

Therefore, switching the roles of x and t with those of x_0 and t_0, respectively,

$$u(x, t) = x + \left(t - \frac{x}{c}\right)^2, \quad x - ct < 0.$$

In conclusion, the solution to the initial boundary value problem is

$$u(x, t) = \begin{cases} x + \left(t - \frac{x}{c}\right)^2 & \text{if } x - ct < 0 \\ x & \text{if } x - ct > 0 \end{cases}.$$