Exercise 15

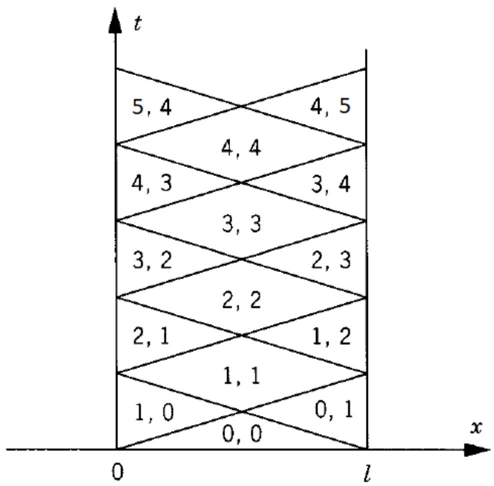
Derive the solution of the wave equation in a finite interval with inhomogeneous boundary conditions $v(0,t) = h(t)$, $v(l,t) = k(t)$, and with $\phi = \psi = f = 0$.

Solution

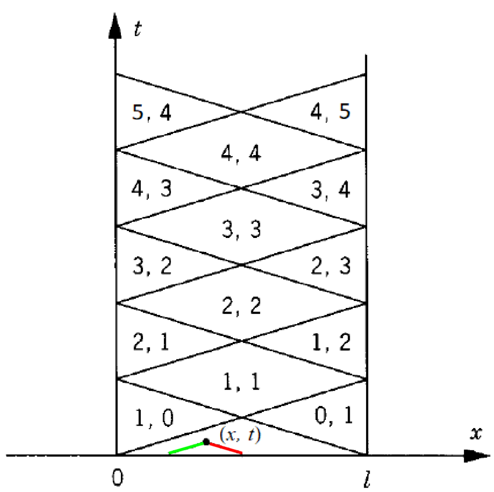
Here we will solve the initial boundary value problem,

$$\begin{aligned} v_{tt} &= c^2 v_{xx} \qquad 0 < x < l,\ t > 0 \\ v(x,0) &= 0 \qquad v(0,t) = h(t) \\ v_t(x,0) &= 0 \qquad v(l,t) = k(t), \end{aligned}$$

using the method of reflection. Depending what point (x,t) is chosen, a different number of reflections can occur, resulting in different solutions. We will determine v for points within each of the diamond-shaped regions illustrated below.



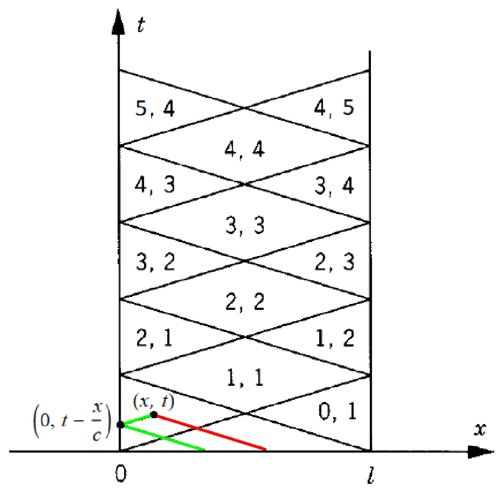
Diamond $\langle 0, 0\rangle$



Neither the green path nor the red path touches the boundaries where v is nonzero, so v is zero.

$$v(x,t) = 0$$

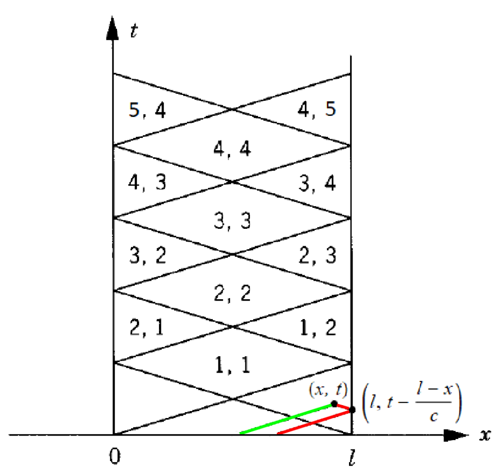
Diamond $\langle 1, 0 \rangle$



There are no reflections on the way to $\left(0, t - \frac{x}{c}\right)$, so there are no minus signs in the coefficient.

$$\begin{aligned} v(x,t) &= (-1)^0 v\left(0, t - \frac{x}{c}\right) \\ &= h\left(t - \frac{x}{c}\right) \end{aligned}$$

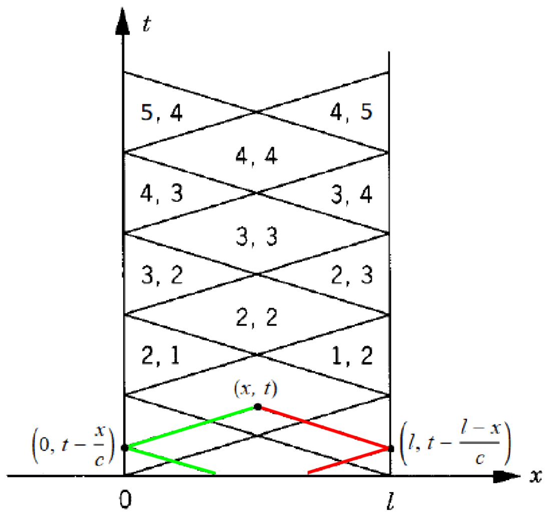
Diamond $\langle 0, 1 \rangle$



There are no reflections on the way to $\left(l, t - \frac{l-x}{c}\right)$, so there are no minus signs in the coefficient.

$$\begin{aligned} v(x,t) &= (-1)^0 v\left(l, t - \frac{l-x}{c}\right) \\ &= k\left(t - \frac{l-x}{c}\right) \end{aligned}$$

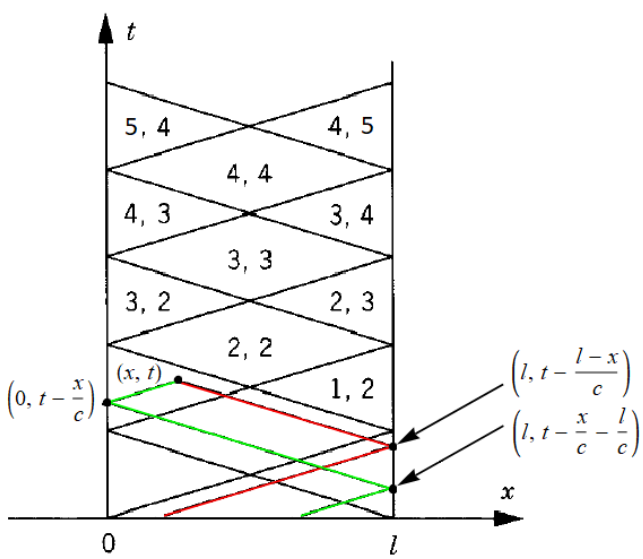
Diamond $\langle 1, 1\rangle$



There are no reflections on the way to $\left(0, t - \frac{x}{c}\right)$ and $\left(l, t - \frac{l-x}{c}\right)$, so there are no minus signs in the coefficients.

$$\begin{aligned} v(x,t) &= (-1)^0 v\left(0, t - \frac{x}{c}\right) + (-1)^0 v\left(l, t - \frac{l-x}{c}\right) \\ &= h\left(t - \frac{x}{c}\right) + k\left(t - \frac{l-x}{c}\right) \end{aligned}$$

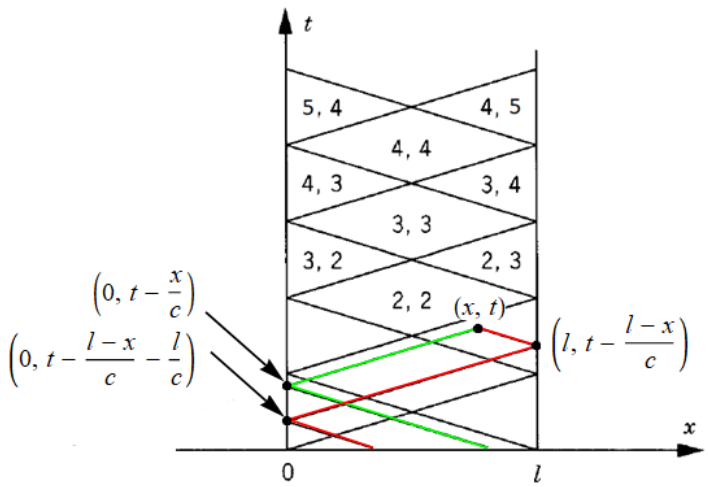
Diamond $\langle 2, 1\rangle$



The number of minus signs in each coefficient is equal to the number of reflections on the way to a particular point.

$$\begin{aligned} v(x,t) &= (-1)^0 v\left(0, t - \frac{x}{c}\right) + (-1)^0 v\left(l, t - \frac{l-x}{c}\right) + (-1)^1 v\left(l, t - \frac{x}{c} - \frac{l}{c}\right) \\ &= h\left(t - \frac{x}{c}\right) + k\left(t - \frac{l-x}{c}\right) - k\left(t - \frac{x}{c} - \frac{l}{c}\right) \end{aligned}$$

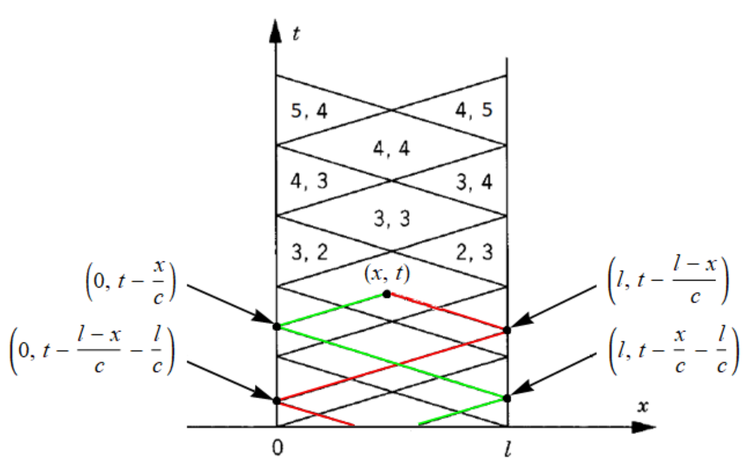
Diamond $\langle 1, 2\rangle$



The number of minus signs in each coefficient is equal to the number of reflections on the way to a particular point.

$$\begin{aligned} v(x,t) &= (-1)^0 v\left(0, t - \frac{x}{c}\right) + (-1)^0 v\left(l, t - \frac{l-x}{c}\right) + (-1)^1 v\left(0, t - \frac{l-x}{c} - \frac{l}{c}\right) \\ &= h\left(t - \frac{x}{c}\right) + k\left(t - \frac{l-x}{c}\right) - h\left(t - \frac{l-x}{c} - \frac{l}{c}\right) \end{aligned}$$

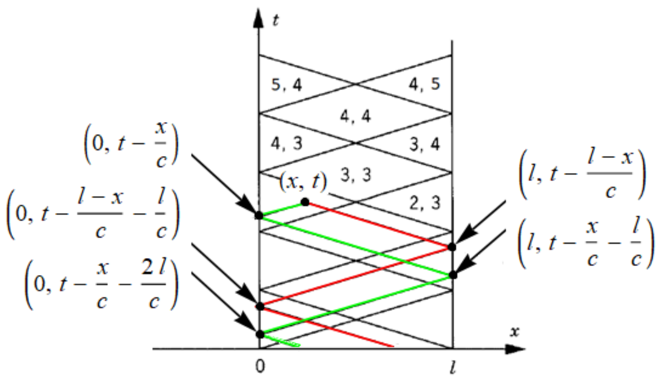
Diamond $\langle 2, 2\rangle$



The number of minus signs in each coefficient is equal to the number of reflections on the way to a particular point.

$$\begin{aligned} v(x,t) &= (-1)^0 v\left(0, t-\frac{x}{c}\right) + (-1)^0 v\left(l, t-\frac{l-x}{c}\right) + (-1)^1 v\left(0, t-\frac{l-x}{c}-\frac{l}{c}\right) + (-1)^1 v\left(l, t-\frac{x}{c}-\frac{l}{c}\right) \\ &= h\left(t-\frac{x}{c}\right) + k\left(t-\frac{l-x}{c}\right) - h\left(t-\frac{l-x}{c}-\frac{l}{c}\right) - k\left(t-\frac{x}{c}-\frac{l}{c}\right) \end{aligned}$$

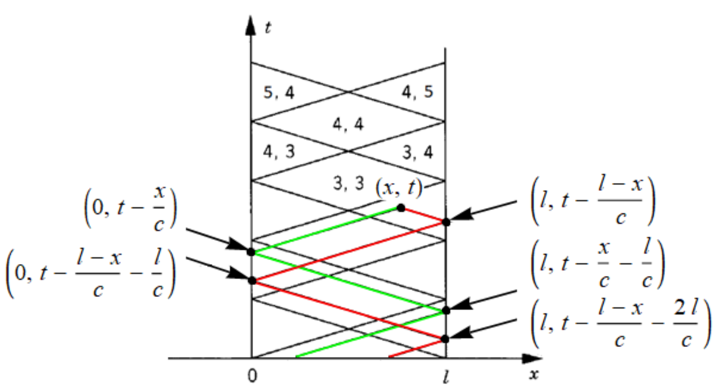
Diamond $\langle 3, 2\rangle$



The number of minus signs in each coefficient is equal to the number of reflections on the way to a particular point.

$$\begin{aligned} v(x,t) &= (-1)^0 v\left(0, t-\frac{x}{c}\right) + (-1)^0 v\left(l, t-\frac{l-x}{c}\right) + (-1)^1 v\left(0, t-\frac{l-x}{c}-\frac{l}{c}\right) \\ &\qquad + (-1)^1 v\left(l, t-\frac{x}{c}-\frac{l}{c}\right) + (-1)^2 v\left(0, t-\frac{x}{c}-\frac{2l}{c}\right) \\ &= h\left(t-\frac{x}{c}\right) + k\left(t-\frac{l-x}{c}\right) - h\left(t-\frac{l-x}{c}-\frac{l}{c}\right) - k\left(t-\frac{x}{c}-\frac{l}{c}\right) + h\left(t-\frac{x}{c}-\frac{2l}{c}\right) \end{aligned}$$

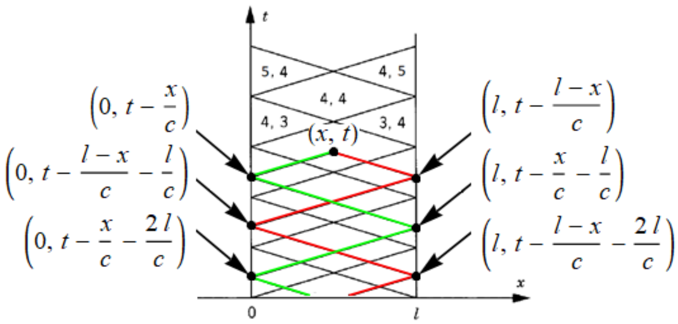
Diamond $\langle 2, 3\rangle$



The number of minus signs in each coefficient is equal to the number of reflections on the way to a particular point.

$$\begin{aligned}
v(x,t) &= (-1)^0 v\left(0, t - \frac{x}{c}\right) + (-1)^0 v\left(l, t - \frac{l-x}{c}\right) + (-1)^1 v\left(0, t - \frac{l-x}{c} - \frac{l}{c}\right) \\
&\qquad + (-1)^1 v\left(l, t - \frac{x}{c} - \frac{l}{c}\right) + (-1)^2 v\left(l, t - \frac{l-x}{c} - \frac{2l}{c}\right) \\
&= h\left(t - \frac{x}{c}\right) + k\left(t - \frac{l-x}{c}\right) - h\left(t - \frac{l-x}{c} - \frac{l}{c}\right) - k\left(t - \frac{x}{c} - \frac{l}{c}\right) + k\left(t - \frac{l-x}{c} - \frac{2l}{c}\right)
\end{aligned}$$

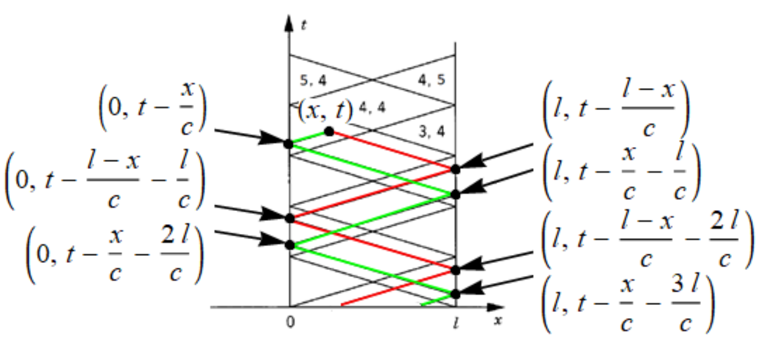
Diamond $\langle 3, 3\rangle$



The number of minus signs in each coefficient is equal to the number of reflections on the way to a particular point.

$$\begin{aligned}
v(x,t) &= (-1)^0 v\left(0, t - \frac{x}{c}\right) + (-1)^0 v\left(l, t - \frac{l-x}{c}\right) + (-1)^1 v\left(0, t - \frac{l-x}{c} - \frac{l}{c}\right) \\
&\qquad + (-1)^1 v\left(l, t - \frac{x}{c} - \frac{l}{c}\right) + (-1)^2 v\left(0, t - \frac{x}{c} - \frac{2l}{c}\right) + (-1)^2 v\left(l, t - \frac{l-x}{c} - \frac{2l}{c}\right) \\
&= h\left(t - \frac{x}{c}\right) + k\left(t - \frac{l-x}{c}\right) - h\left(t - \frac{l-x}{c} - \frac{l}{c}\right) - k\left(t - \frac{x}{c} - \frac{l}{c}\right) + h\left(t - \frac{x}{c} - \frac{2l}{c}\right) \\
&\qquad + k\left(t - \frac{l-x}{c} - \frac{2l}{c}\right)
\end{aligned}$$

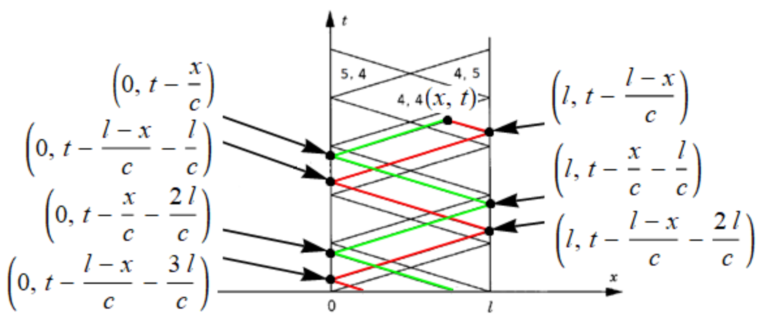
Diamond $\langle 4, 3\rangle$



The number of minus signs in each coefficient is equal to the number of reflections on the way to a particular point.

$$\begin{aligned}
v(x,t) &= (-1)^0 v\left(0, t-\frac{x}{c}\right) + (-1)^0 v\left(l, t-\frac{l-x}{c}\right) + (-1)^1 v\left(0, t-\frac{l-x}{c}-\frac{l}{c}\right) \\
&\quad + (-1)^1 v\left(l, t-\frac{x}{c}-\frac{l}{c}\right) + (-1)^2 v\left(0, t-\frac{x}{c}-\frac{2l}{c}\right) + (-1)^2 v\left(l, t-\frac{l-x}{c}-\frac{2l}{c}\right) \\
&\quad + (-1)^3 v\left(l, t-\frac{x}{c}-\frac{3l}{c}\right) \\
&= h\left(t-\frac{x}{c}\right) + k\left(t-\frac{l-x}{c}\right) - h\left(t-\frac{l-x}{c}-\frac{l}{c}\right) - k\left(t-\frac{x}{c}-\frac{l}{c}\right) + h\left(t-\frac{x}{c}-\frac{2l}{c}\right) \\
&\quad + k\left(t-\frac{l-x}{c}-\frac{2l}{c}\right) - k\left(t-\frac{x}{c}-\frac{3l}{c}\right)
\end{aligned}$$

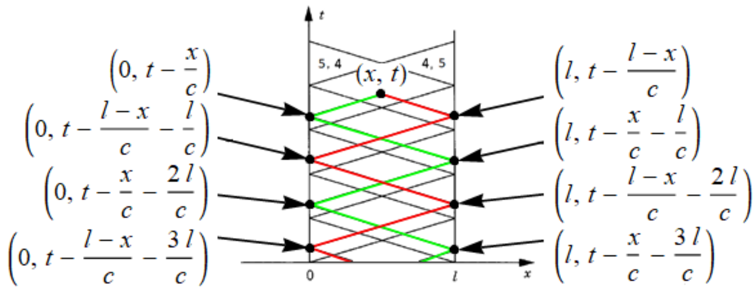
Diamond $\langle 3,4\rangle$



The number of minus signs in each coefficient is equal to the number of reflections on the way to a particular point.

$$\begin{aligned}
v(x,t) &= (-1)^0 v\left(0, t-\frac{x}{c}\right) + (-1)^0 v\left(l, t-\frac{l-x}{c}\right) + (-1)^1 v\left(0, t-\frac{l-x}{c}-\frac{l}{c}\right) \\
&\quad + (-1)^1 v\left(l, t-\frac{x}{c}-\frac{l}{c}\right) + (-1)^2 v\left(0, t-\frac{x}{c}-\frac{2l}{c}\right) + (-1)^2 v\left(l, t-\frac{l-x}{c}-\frac{2l}{c}\right) \\
&\quad + (-1)^3 v\left(0, t-\frac{l-x}{c}-\frac{3l}{c}\right) \\
&= h\left(t-\frac{x}{c}\right) + k\left(t-\frac{l-x}{c}\right) - h\left(t-\frac{l-x}{c}-\frac{l}{c}\right) - k\left(t-\frac{x}{c}-\frac{l}{c}\right) + h\left(t-\frac{x}{c}-\frac{2l}{c}\right) \\
&\quad + k\left(t-\frac{l-x}{c}-\frac{2l}{c}\right) - h\left(t-\frac{l-x}{c}-\frac{3l}{c}\right)
\end{aligned}$$

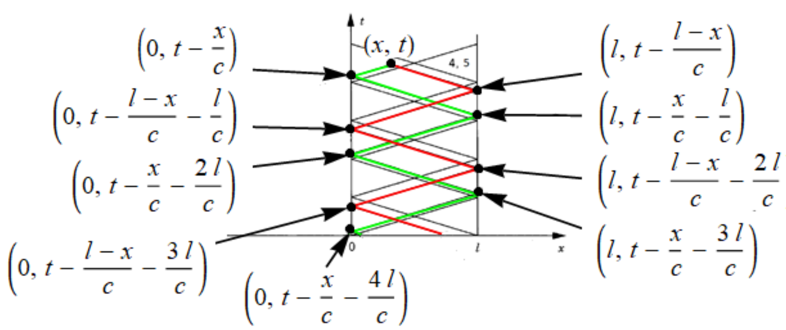
Diamond $\langle 4, 4\rangle$



The number of minus signs in each coefficient is equal to the number of reflections on the way to a particular point.

$$\begin{aligned}
v(x,t) &= (-1)^0 v\left(0, t - \frac{x}{c}\right) + (-1)^0 v\left(l, t - \frac{l-x}{c}\right) + (-1)^1 v\left(0, t - \frac{l-x}{c} - \frac{l}{c}\right) \\
&\quad + (-1)^1 v\left(l, t - \frac{x}{c} - \frac{l}{c}\right) + (-1)^2 v\left(0, t - \frac{x}{c} - \frac{2l}{c}\right) + (-1)^2 v\left(l, t - \frac{l-x}{c} - \frac{2l}{c}\right) \\
&\quad + (-1)^3 v\left(0, t - \frac{l-x}{c} - \frac{3l}{c}\right) + (-1)^3 v\left(l, t - \frac{x}{c} - \frac{3l}{c}\right) \\
&= h\left(t - \frac{x}{c}\right) + k\left(t - \frac{l-x}{c}\right) - h\left(t - \frac{l-x}{c} - \frac{l}{c}\right) - k\left(t - \frac{x}{c} - \frac{l}{c}\right) + h\left(t - \frac{x}{c} - \frac{2l}{c}\right) \\
&\quad + k\left(t - \frac{l-x}{c} - \frac{2l}{c}\right) - h\left(t - \frac{l-x}{c} - \frac{3l}{c}\right) - k\left(t - \frac{x}{c} - \frac{3l}{c}\right)
\end{aligned}$$

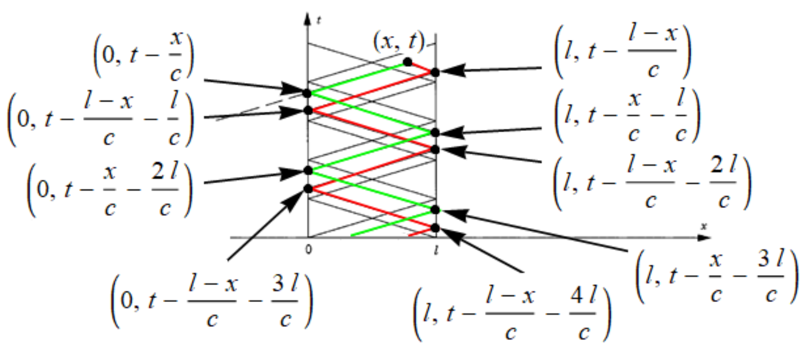
Diamond $\langle 5, 4\rangle$



The number of minus signs in each coefficient is equal to the number of reflections on the way to a particular point.

$$
\begin{aligned}
v(x,t) &= (-1)^0 v\left(0, t-\frac{x}{c}\right) + (-1)^0 v\left(l, t-\frac{l-x}{c}\right) + (-1)^1 v\left(0, t-\frac{l-x}{c}-\frac{l}{c}\right) \\
&\quad + (-1)^1 v\left(l, t-\frac{x}{c}-\frac{l}{c}\right) + (-1)^2 v\left(0, t-\frac{x}{c}-\frac{2l}{c}\right) + (-1)^2 v\left(l, t-\frac{l-x}{c}-\frac{2l}{c}\right) \\
&\quad + (-1)^3 v\left(0, t-\frac{l-x}{c}-\frac{3l}{c}\right) + (-1)^3 v\left(l, t-\frac{x}{c}-\frac{3l}{c}\right) \\
&\quad + (-1)^4 v\left(0, t-\frac{x}{c}-\frac{4l}{c}\right) \\
&= h\left(t-\frac{x}{c}\right) + k\left(t-\frac{l-x}{c}\right) - h\left(t-\frac{l-x}{c}-\frac{l}{c}\right) - k\left(t-\frac{x}{c}-\frac{l}{c}\right) + h\left(t-\frac{x}{c}-\frac{2l}{c}\right) \\
&\quad + k\left(t-\frac{l-x}{c}-\frac{2l}{c}\right) - h\left(t-\frac{l-x}{c}-\frac{3l}{c}\right) - k\left(t-\frac{x}{c}-\frac{3l}{c}\right) + h\left(t-\frac{x}{c}-\frac{4l}{c}\right)
\end{aligned}
$$

Diamond $\langle 4, 5\rangle$



The number of minus signs in each coefficient is equal to the number of reflections on the way to a particular point.

$$
\begin{aligned}
v(x,t) &= (-1)^0 v\left(0, t-\frac{x}{c}\right) + (-1)^0 v\left(l, t-\frac{l-x}{c}\right) + (-1)^1 v\left(0, t-\frac{l-x}{c}-\frac{l}{c}\right) \\
&\quad + (-1)^1 v\left(l, t-\frac{x}{c}-\frac{l}{c}\right) + (-1)^2 v\left(0, t-\frac{x}{c}-\frac{2l}{c}\right) + (-1)^2 v\left(l, t-\frac{l-x}{c}-\frac{2l}{c}\right) \\
&\quad + (-1)^3 v\left(0, t-\frac{l-x}{c}-\frac{3l}{c}\right) + (-1)^3 v\left(l, t-\frac{x}{c}-\frac{3l}{c}\right) \\
&\quad + (-1)^4 v\left(l, t-\frac{l-x}{c}-\frac{4l}{c}\right) \\
&= h\left(t-\frac{x}{c}\right) + k\left(t-\frac{l-x}{c}\right) - h\left(t-\frac{l-x}{c}-\frac{l}{c}\right) - k\left(t-\frac{x}{c}-\frac{l}{c}\right) + h\left(t-\frac{x}{c}-\frac{2l}{c}\right) \\
&\quad + k\left(t-\frac{l-x}{c}-\frac{2l}{c}\right) - h\left(t-\frac{l-x}{c}-\frac{3l}{c}\right) - k\left(t-\frac{x}{c}-\frac{3l}{c}\right) + k\left(t-\frac{l-x}{c}-\frac{4l}{c}\right)
\end{aligned}
$$

Therefore,

$$v(x,t)=\begin{cases}
0 & \text{in } \langle 0,0\rangle \\
h\left(t-\frac{x}{c}\right) & \text{in } \langle 1,0\rangle \\
k\left(t-\frac{l-x}{c}\right) & \text{in } \langle 0,1\rangle \\
h\left(t-\frac{x}{c}\right)+k\left(t-\frac{l-x}{c}\right) & \text{in } \langle 1,1\rangle \\
h\left(t-\frac{x}{c}\right)+k\left(t-\frac{l-x}{c}\right)-k\left(t-\frac{x}{c}-\frac{l}{c}\right) & \text{in } \langle 2,1\rangle \\
h\left(t-\frac{x}{c}\right)+k\left(t-\frac{l-x}{c}\right)-h\left(t-\frac{l-x}{c}-\frac{l}{c}\right) & \text{in } \langle 1,2\rangle \\
h\left(t-\frac{x}{c}\right)-h\left(t-\frac{l-x}{c}-\frac{l}{c}\right)-k\left(t-\frac{x}{c}-\frac{l}{c}\right)+k\left(t-\frac{l-x}{c}\right) & \text{in } \langle 2,2\rangle \\
\sum_{n=0}^{1}h\left(t-\frac{x}{c}-\frac{2nl}{c}\right)-h\left(t-\frac{l-x}{c}-\frac{l}{c}\right)-k\left(t-\frac{x}{c}-\frac{l}{c}\right)+k\left(t-\frac{l-x}{c}\right) & \text{in } \langle 3,2\rangle \\
h\left(t-\frac{x}{c}\right)-h\left(t-\frac{l-x}{c}-\frac{l}{c}\right)-k\left(t-\frac{x}{c}-\frac{l}{c}\right)+\sum_{n=0}^{1}k\left(t-\frac{l-x}{c}-\frac{2nl}{c}\right) & \text{in } \langle 2,3\rangle \\
\sum_{n=0}^{1}h\left(t-\frac{x}{c}-\frac{2nl}{c}\right)-h\left(t-\frac{l-x}{c}-\frac{l}{c}\right)-k\left(t-\frac{x}{c}-\frac{l}{c}\right)+\sum_{n=0}^{1}k\left(t-\frac{l-x}{c}-\frac{2nl}{c}\right) & \text{in } \langle 3,3\rangle \\
\sum_{n=0}^{1}h\left(t-\frac{x}{c}-\frac{2nl}{c}\right)-h\left(t-\frac{l-x}{c}-\frac{l}{c}\right)-\sum_{n=0}^{1}k\left(t-\frac{x}{c}-\frac{(2n+1)l}{c}\right)+\sum_{n=0}^{1}k\left(t-\frac{l-x}{c}-\frac{2nl}{c}\right) & \text{in } \langle 4,3\rangle \\
\sum_{n=0}^{1}h\left(t-\frac{x}{c}-\frac{2nl}{c}\right)-\sum_{n=0}^{1}h\left(t-\frac{l-x}{c}-\frac{(2n+1)l}{c}\right)-k\left(t-\frac{x}{c}-\frac{l}{c}\right)+\sum_{n=0}^{1}k\left(t-\frac{l-x}{c}-\frac{2nl}{c}\right) & \text{in } \langle 3,4\rangle \\
\sum_{n=0}^{1}h\left(t-\frac{x}{c}-\frac{2nl}{c}\right)-\sum_{n=0}^{1}h\left(t-\frac{l-x}{c}-\frac{(2n+1)l}{c}\right)-\sum_{n=0}^{1}k\left(t-\frac{x}{c}-\frac{(2n+1)l}{c}\right)+\sum_{n=0}^{1}k\left(t-\frac{l-x}{c}-\frac{2nl}{c}\right) & \text{in } \langle 4,4\rangle \\
\sum_{n=0}^{2}h\left(t-\frac{x}{c}-\frac{2nl}{c}\right)-\sum_{n=0}^{1}h\left(t-\frac{l-x}{c}-\frac{(2n+1)l}{c}\right)-\sum_{n=0}^{1}k\left(t-\frac{x}{c}-\frac{(2n+1)l}{c}\right)+\sum_{n=0}^{1}k\left(t-\frac{l-x}{c}-\frac{2nl}{c}\right) & \text{in } \langle 5,4\rangle \\
\sum_{n=0}^{1}h\left(t-\frac{x}{c}-\frac{2nl}{c}\right)-\sum_{n=0}^{1}h\left(t-\frac{l-x}{c}-\frac{(2n+1)l}{c}\right)-\sum_{n=0}^{1}k\left(t-\frac{x}{c}-\frac{(2n+1)l}{c}\right)+\sum_{n=0}^{2}k\left(t-\frac{l-x}{c}-\frac{2nl}{c}\right) & \text{in } \langle 4,5\rangle \\
\vdots & \vdots
\end{cases}.$$

There are a few things one should know about this solution. First, each formula satisfies the wave equation. Secondly, the formula for the one region in contact with $t = 0$, diamond $\langle 0, 0\rangle$, satisfies the initial conditions, $v(x, 0) = 0$ and $v_t(x, 0) = 0$. Thirdly, the formulas for all of the regions in contact with $x = 0$, namely diamonds $\langle 1, 0\rangle$, $\langle 2, 1\rangle$, $\langle 3, 2\rangle$, $\langle 4, 3\rangle$, and $\langle 5, 4\rangle$, satisfy the boundary condition $v(0, t) = h(t)$. Finally, the formulas for all of the regions in contact with $x = l$, namely diamonds $\langle 0, 1\rangle$, $\langle 1, 2\rangle$, $\langle 2, 3\rangle$, $\langle 3, 4\rangle$, and $\langle 4, 5\rangle$, satisfy the boundary condition $v(l, t) = k(t)$.