

Exercise 2

Solve $u_{tt} = c^2 u_{xx} + e^{ax}$, $u(x, 0) = 0$, $u_t(x, 0) = 0$.

Solution

Solution by Operator Factorization

Bring $c^2 u_{xx}$ to the other side.

$$u_{tt} - c^2 u_{xx} = e^{ax}$$

Write the left side as an operator acting on u .

$$(\partial_t^2 - c^2 \partial_x^2)u = e^{ax}$$

The operator is a difference of squares, so it can be factored.

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = e^{ax}$$

Let

$$v = (\partial_t - c\partial_x)u$$

so that the PDE becomes

$$(\partial_t + c\partial_x)v = e^{ax}.$$

The second-order PDE we started with has thus been reduced to the following system of first-order PDEs that can be solved with the method of characteristics.

$$u_t - cu_x = v \tag{1}$$

$$v_t + cv_x = e^{ax} \tag{2}$$

We will solve the second one for v first, and once that is known, the first equation for u will be solved. For a function of two variables $\phi = \phi(x, t)$, its differential is defined as

$$d\phi = \frac{\partial\phi}{\partial t} dt + \frac{\partial\phi}{\partial x} dx.$$

If we divide both sides by dt , then we get the relationship between the ordinary derivative of ϕ and its partial derivatives.

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} \frac{dx}{dt} \tag{3}$$

Comparing this with equation (2), we see that along the curves in the xt -plane that satisfy

$$\frac{dx}{dt} = c, \tag{4}$$

the PDE for $v(x, t)$ reduces to an ODE.

$$\frac{dv}{dt} = e^{ax} \tag{5}$$

Because c is a constant, equation (4) can be solved by integrating both sides with respect to t .

$$x = ct + \xi, \tag{6}$$

where ξ is a characteristic coordinate. Substitute this expression for x into equation (5) to obtain an ODE that only involves t (ξ is regarded as a constant).

$$\frac{dv}{dt} = e^{a(ct+\xi)}$$

Distribute a and split the exponential function into two.

$$\frac{dv}{dt} = e^{act} e^{a\xi}$$

Integrate both sides with respect to t .

$$v(\xi, t) = \frac{e^{a\xi}}{ac} e^{act} + f(\xi),$$

where f is an arbitrary function of the characteristic coordinate ξ . In order to write v in terms of x and t , solve equation (6) for ξ .

$$x = ct + \xi \quad \rightarrow \quad \xi = x - ct$$

Hence,

$$\begin{aligned} v(x, t) &= \frac{e^{a(x-ct)}}{ac} e^{act} + f(x - ct) \\ &= \frac{e^{ax}}{ac} + f(x - ct). \end{aligned}$$

As a result, equation (1) becomes

$$u_t - cu_x = \frac{e^{ax}}{ac} + f(x - ct).$$

Comparing this equation with equation (3), we see that along the curves in the xt -plane that satisfy

$$\frac{dx}{dt} = -c, \tag{7}$$

the PDE for $u(x, t)$ reduces to an ODE.

$$\frac{du}{dt} = \frac{e^{ax}}{ac} + f(x - ct) \tag{8}$$

Because c is a constant, equation (7) can be solved by integrating both sides with respect to t .

$$x = -ct + \eta, \tag{9}$$

where η is another characteristic coordinate. Substitute this expression for x into equation (8) to obtain an ODE that only involves t (η is regarded as a constant).

$$\begin{aligned} \frac{du}{dt} &= \frac{e^{a(-ct+\eta)}}{ac} + f(-ct + \eta - ct) \\ \frac{du}{dt} &= \frac{e^{a\eta}}{ac} e^{-act} + f(\eta - 2ct) \end{aligned}$$

Integrate both sides with respect to t .

$$u(\eta, t) = -\frac{e^{a\eta}}{a^2c^2}e^{-act} + \int^t f(\eta - 2cs) ds + g(\eta),$$

where g is an arbitrary function of the characteristic coordinate η . The integral of an arbitrary function is another arbitrary function.

$$u(\eta, t) = -\frac{e^{a\eta}}{a^2c^2}e^{-act} + F(\eta - 2ct) + g(\eta),$$

In order to write u in terms of x and t , solve equation (9) for η .

$$x = -ct + \eta \quad \rightarrow \quad \eta = x + ct$$

Hence,

$$\begin{aligned} u(x, t) &= -\frac{e^{a(x+ct)}}{a^2c^2}e^{-act} + F(x + ct - 2ct) + g(x + ct) \\ &= -\frac{e^{ax}}{a^2c^2} + F(x - ct) + g(x + ct). \end{aligned}$$

This is the general solution to $u_{tt} = c^2u_{xx} + e^{ax}$. If we apply the two initial conditions, we can determine F and g . Before doing so, take a derivative of the solution with respect to t .

$$u_t(x, t) = -cF'(x - ct) + cg'(x + ct)$$

From the initial conditions we obtain the following system of equations.

$$\begin{aligned} u(x, 0) &= -\frac{e^{ax}}{a^2c^2} + F(x) + g(x) = 0 \\ u_t(x, 0) &= -cF'(x) + cg'(x) = 0 \end{aligned}$$

Even though this system is in terms of x , it's really in terms of w , where w is any expression we choose.

$$\begin{aligned} -\frac{e^{aw}}{a^2c^2} + F(w) + g(w) &= 0 \\ -cF'(w) + cg'(w) &= 0 \end{aligned}$$

Differentiating both sides of the first equation with respect to w , we get

$$-\frac{e^{aw}}{ac^2} + F'(w) + g'(w) = 0 \quad \rightarrow \quad g'(w) = \frac{e^{aw}}{ac^2} - F'(w).$$

Plug this expression for $g'(w)$ into the second equation.

$$-cF'(w) + c \left[\frac{e^{aw}}{ac^2} - F'(w) \right] = 0 \quad \rightarrow \quad -2cF'(w) + \frac{e^{aw}}{ac} = 0 \quad \rightarrow \quad F'(w) = \frac{e^{aw}}{2ac^2}.$$

Solve for $F(w)$ and obtain an expression for $F(x - ct)$.

$$F(w) = \frac{e^{aw}}{2a^2c^2} + C_1 \quad \Rightarrow \quad F(x - ct) = \frac{e^{a(x-ct)}}{2a^2c^2} + C_1$$

Use the first equation to solve for $g(w)$ and obtain an expression for $g(x + ct)$.

$$\begin{aligned} g(w) &= \frac{e^{aw}}{a^2c^2} - F(w) \\ &= \frac{e^{aw}}{a^2c^2} - \frac{e^{aw}}{2a^2c^2} - C_1 \\ &= \frac{e^{aw}}{2a^2c^2} - C_1 \quad \Rightarrow \quad g(x + ct) = \frac{e^{a(x+ct)}}{2a^2c^2} - C_1 \end{aligned}$$

The general solution for $u(x, t)$ becomes

$$\begin{aligned} u(x, t) &= -\frac{e^{ax}}{a^2c^2} + F(x - ct) + g(x + ct) \\ &= -\frac{e^{ax}}{a^2c^2} + \frac{e^{a(x-ct)}}{2a^2c^2} + \cancel{C_1} + \frac{e^{a(x+ct)}}{2a^2c^2} - \cancel{C_1} \\ &= \frac{e^{ax}}{a^2c^2} \left(-1 + \frac{1}{2}e^{-act} + \frac{1}{2}e^{act} \right) \\ &= \frac{e^{ax}}{a^2c^2} (-1 + \cosh act) \end{aligned}$$

Therefore,

$$u(x, t) = \frac{e^{ax}}{a^2c^2} (\cosh act - 1).$$

Solution by the Method of Characteristics

Bring $c^2 u_{xx}$ to the left side of the PDE.

$$u_{tt} - c^2 u_{xx} = e^{ax}$$

Comparing this with the general form of a second-order PDE,

$$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G,$$

we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = e^{ax}$. The characteristic equations for a second-order PDE are given by

$$\frac{dx}{dt} = \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC}),$$

the solutions of which are known as the characteristics. Since $B^2 - 4AC = 4c^2 > 0$, the PDE is hyperbolic, so the solutions to these equations are two real and distinct families of characteristic curves in the xt -plane.

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2}(\pm\sqrt{4c^2}) \\ \frac{dx}{dt} &= \frac{1}{2}(\pm 2c) \\ \frac{dx}{dt} &= c \quad \text{or} \quad \frac{dx}{dt} = -c\end{aligned}$$

Integrate both sides of each equation with respect to t .

$$x = ct + C_2 \quad \text{or} \quad x = -ct + C_3$$

Now make the substitutions,

$$\begin{aligned}\xi &= x - ct = C_2 \\ \eta &= x + ct = C_3,\end{aligned}$$

so that the PDE takes the simplest form. The aim is to write u_{tt} , u_{xx} , and e^{ax} in terms of the new variables, ξ and η . Solving these two equations for x and t with elimination gives

$$\begin{aligned}x &= \frac{1}{2}(\eta + \xi) \\ t &= \frac{1}{2c}(\eta - \xi).\end{aligned}$$

Use the chain rule to write the old derivatives in terms of the new variables.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t} = u_\xi(-c) + u_\eta(c) = c(u_\eta - u_\xi) \\ \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = u_\xi(1) + u_\eta(1) = u_\xi + u_\eta\end{aligned}$$

Find the second derivatives by using the chain rule again.

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c \frac{\partial}{\partial t}(u_\eta - u_\xi) = c \left(\frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta} \right) (u_\eta - u_\xi) = c \left[\frac{\partial \xi}{\partial t} \frac{\partial}{\partial \xi} (u_\eta - u_\xi) + \frac{\partial \eta}{\partial t} \frac{\partial}{\partial \eta} (u_\eta - u_\xi) \right] \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(u_\xi + u_\eta) = \left(\frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} \right) (u_\xi + u_\eta) = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} (u_\xi + u_\eta) + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} (u_\xi + u_\eta)\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c[(-c)(u_{\xi\eta} - u_{\xi\xi}) + (c)(u_{\eta\eta} - u_{\xi\eta})] = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \\ \frac{\partial^2 u}{\partial x^2} &= (1)(u_{\xi\xi} + u_{\xi\eta}) + (1)(u_{\xi\eta} + u_{\eta\eta}) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.\end{aligned}$$

Substituting these expressions into the PDE, $u_{tt} - c^2 u_{xx} = e^{ax}$, we obtain

$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) - c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = \exp\left[\frac{a}{2}(\eta + \xi)\right].$$

Simplify the left side.

$$-4c^2 u_{\xi\eta} = \exp\left[\frac{a}{2}(\eta + \xi)\right]$$

Divide both sides by $-4c^2$.

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = -\frac{1}{4c^2} \exp\left[\frac{a}{2}(\eta + \xi)\right]$$

This is known as the first canonical form of the PDE. Integrate both sides of it partially with respect to η .

$$\int^{\eta} \frac{\partial^2 u}{\partial \xi \partial \eta} \Big|_{\eta=s} ds = \int^{\eta} -\frac{1}{4c^2} \exp\left[\frac{a}{2}(s + \xi)\right] ds + f(\xi),$$

where f is an arbitrary function of ξ .

$$\begin{aligned}\frac{\partial u}{\partial \xi} &= -\frac{1}{4c^2} \frac{2}{a} \exp\left[\frac{a}{2}(s + \xi)\right] \Big|_{\eta=s}^{\eta} + f(\xi) \\ \frac{\partial u}{\partial \xi} &= -\frac{1}{2ac^2} \exp\left[\frac{a}{2}(\eta + \xi)\right] + f(\xi)\end{aligned}$$

Now integrate both sides partially with respect to ξ .

$$\int^{\xi} \frac{\partial u}{\partial \xi} \Big|_{\xi=s} ds = \int^{\xi} \left\{ -\frac{1}{2ac^2} \exp\left[\frac{a}{2}(\eta + s)\right] + f(s) \right\} ds + g(\eta),$$

where g is an arbitrary function of η .

$$\begin{aligned}u(\xi, \eta) &= \left\{ -\frac{1}{2ac^2} \frac{2}{a} \exp\left[\frac{a}{2}(\eta + s)\right] + F(s) \right\} \Big|_{\xi=s}^{\xi} + g(\eta) \\ u(\xi, \eta) &= -\frac{1}{a^2 c^2} \exp\left[\frac{a}{2}(\eta + \xi)\right] + F(\xi) + g(\eta)\end{aligned}$$

Since u has been solved for, change back to the original variables, x and t , by substituting the expressions for ξ and η .

$$u(x, t) = -\frac{e^{ax}}{a^2 c^2} + F(x - ct) + g(x + ct)$$

This is the general solution to $u_{tt} = c^2 u_{xx} + e^{ax}$. If we apply the two initial conditions, we can determine F and g . Before doing so, take a derivative of the solution with respect to t .

$$u_t(x, t) = -cF'(x - ct) + cg'(x + ct)$$

From the initial conditions we obtain the following system of equations.

$$\begin{aligned}u(x, 0) &= -\frac{e^{ax}}{a^2c^2} + F(x) + g(x) = 0 \\u_t(x, 0) &= -cF'(x) + cg'(x) = 0\end{aligned}$$

Even though this system is in terms of x , it's really in terms of w , where w is any expression we choose.

$$\begin{aligned}-\frac{e^{aw}}{a^2c^2} + F(w) + g(w) &= 0 \\-cF'(w) + cg'(w) &= 0\end{aligned}$$

Differentiating both sides of the first equation with respect to w , we get

$$-\frac{e^{aw}}{ac^2} + F'(w) + g'(w) = 0 \quad \rightarrow \quad g'(w) = \frac{e^{aw}}{ac^2} - F'(w).$$

Plug this expression for $g'(w)$ into the second equation.

$$-cF'(w) + c \left[\frac{e^{aw}}{ac^2} - F'(w) \right] = 0 \quad \rightarrow \quad -2cF'(w) + \frac{e^{aw}}{ac} = 0 \quad \rightarrow \quad F'(w) = \frac{e^{aw}}{2ac^2}.$$

Solve for $F(w)$ and obtain an expression for $F(x - ct)$.

$$F(w) = \frac{e^{aw}}{2a^2c^2} + C_1 \quad \Rightarrow \quad F(x - ct) = \frac{e^{a(x-ct)}}{2a^2c^2} + C_1$$

Use the first equation to solve for $g(w)$ and obtain an expression for $g(x + ct)$.

$$\begin{aligned}g(w) &= \frac{e^{aw}}{a^2c^2} - F(w) \\&= \frac{e^{aw}}{a^2c^2} - \frac{e^{aw}}{2a^2c^2} - C_1 \\&= \frac{e^{aw}}{2a^2c^2} - C_1 \quad \Rightarrow \quad g(x + ct) = \frac{e^{a(x+ct)}}{2a^2c^2} - C_1\end{aligned}$$

The general solution for $u(x, t)$ becomes

$$\begin{aligned}u(x, t) &= -\frac{e^{ax}}{a^2c^2} + F(x - ct) + g(x + ct) \\&= -\frac{e^{ax}}{a^2c^2} + \frac{e^{a(x-ct)}}{2a^2c^2} + \cancel{C_1} + \frac{e^{a(x+ct)}}{2a^2c^2} - \cancel{C_1} \\&= \frac{e^{ax}}{a^2c^2} \left(-1 + \frac{1}{2}e^{-act} + \frac{1}{2}e^{act} \right) \\&= \frac{e^{ax}}{a^2c^2} (-1 + \cosh act)\end{aligned}$$

Therefore,

$$u(x, t) = \frac{e^{ax}}{a^2c^2} (\cosh act - 1).$$

Solution by Green's Theorem

$$u_{tt} - c^2 u_{xx} = e^{ax}, \quad -\infty < x < \infty, \quad t > 0$$

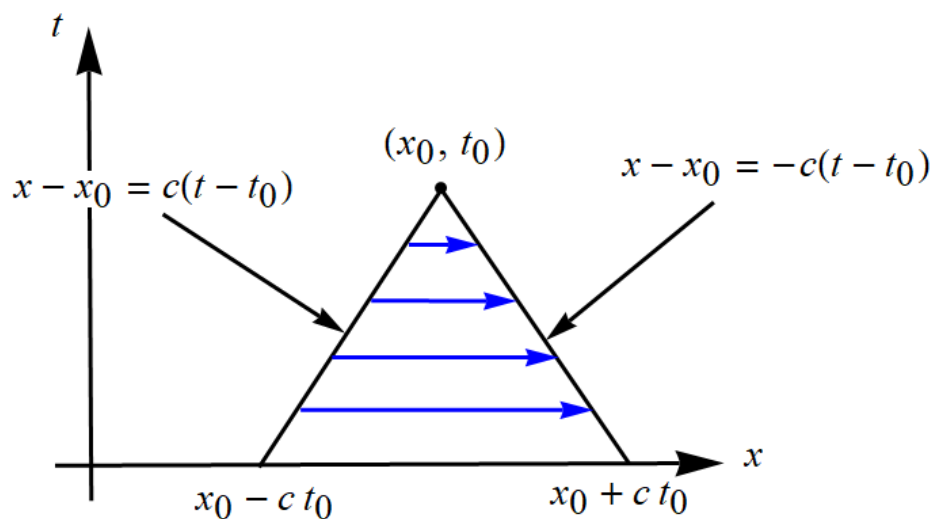
$$u(x, 0) = 0 \quad u_t(x, 0) = 0$$

The characteristics were found to be straight lines, $\xi = x - ct$ and $\eta = x + ct$, with slopes $\pm c$. Suppose (x_0, t_0) is the point in the xt -plane we want to evaluate u at. The equations of the lines going through this point are

$$x - x_0 = c(t - t_0)$$

$$x - x_0 = -c(t - t_0).$$

Integrate both sides of the inhomogeneous wave equation over the triangular domain D enclosed by these lines (from left to right as indicated below).



Write the double integral explicitly on the right side.

$$\iint_D (u_{tt} - c^2 u_{xx}) dA = \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} e^{ax} dx dt$$

Rewrite the left side.

$$-\iint_D \left[\frac{\partial}{\partial x} (c^2 u_x) - \frac{\partial}{\partial t} (u_t) \right] dA = \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} e^{ax} dx dt$$

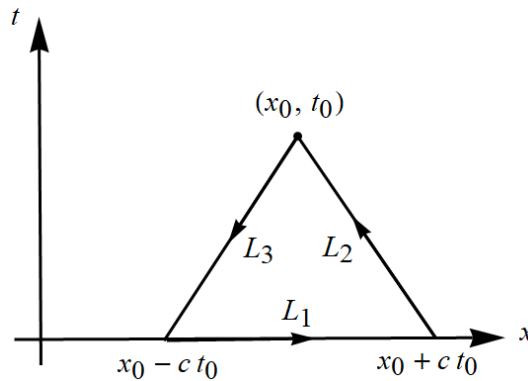
Multiply both sides by -1 .

$$\iint_D \left[\frac{\partial}{\partial x} (c^2 u_x) - \frac{\partial}{\partial t} (u_t) \right] dA = - \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} e^{ax} dx dt$$

Apply Green's theorem (essentially the divergence theorem in two dimensions) to the double integral on the left to turn it into a counterclockwise line integral around the triangle's boundary $\text{bdy } D$.

$$\oint_{\text{bdy } D} (u_t dx + c^2 u_x dt) = - \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} e^{ax} dx dt$$

Let L_1 , L_2 , and L_3 represent the legs of the triangle.



The line integral is the sum of three integrals, one over each leg.

$$\int_{L_1} (u_t dx + c^2 u_x dt) + \int_{L_2} (u_t dx + c^2 u_x dt) + \int_{L_3} (u_t dx + c^2 u_x dt) = - \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} e^{ax} dx dt$$

On L_1	On L_2	On L_3
$t = 0$	$x - x_0 = -c(t - t_0)$	$x - x_0 = c(t - t_0)$
$dt = 0$	$dx = -c dt$	$dx = c dt$

Replace the differentials in the integrals over L_2 and L_3 .

$$\int_{x_0-ct_0}^{x_0+ct_0} u_t(x, 0) dx + \int_{L_2} \left[u_t(-c dt) + c^2 u_x \left(-\frac{dx}{c} \right) \right] + \int_{L_3} \left[u_t(c dt) + c^2 u_x \left(\frac{dx}{c} \right) \right] = - \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} e^{ax} dx dt$$

In this exercise $u_t(x, 0) = 0$, so the integral over L_1 vanishes.

$$-c \int_{L_2} \left(\frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx \right) + c \int_{L_3} \left(\frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx \right) = - \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} e^{ax} dx dt$$

The integrands are how the differential of $u = u(x, t)$ is defined.

$$-c \int_{L_2} du + c \int_{L_3} du = - \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} e^{ax} dx dt$$

Evaluate the integrals on the left side.

$$-c[u(x_0, t_0) - u(x_0 + ct_0, 0)] + c[u(x_0 - ct_0, 0) - u(x_0, t_0)] = - \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} e^{ax} dx dt$$

In this exercise $u(x, 0) = 0$, so $u(x_0 + ct_0, 0) = 0$ and $u(x_0 - ct_0, 0) = 0$.

$$-2cu(x_0, t_0) = - \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} e^{ax} dx dt$$

Divide both sides by $-2c$.

$$u(x_0, t_0) = \frac{1}{2c} \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} e^{ax} dx dt$$

Finally, switch the roles of x and t with those of x_0 and t_0 , respectively.

$$u(x, t) = \frac{1}{2c} \int_0^t \int_{x+c(t_0-t)}^{x-c(t_0-t)} e^{ax_0} dx_0 dt_0$$

Proceed to evaluate the last integral.

$$\begin{aligned} u(x, t) &= \frac{1}{2c} \int_0^t \int_{x-c(t-t_0)}^{x+c(t-t_0)} e^{ax_0} dx_0 dt_0 \\ &= \frac{1}{2c} \int_0^t \frac{e^{ax_0}}{a} \Big|_{x-c(t-t_0)}^{x+c(t-t_0)} dt_0 \\ &= \frac{1}{2ac} \int_0^t \{e^{a[x+c(t-t_0)]} - e^{a[x-c(t-t_0)]}\} dt_0 \\ &= \frac{1}{2ac} \left[e^{a(x+ct)} \int_0^t e^{-act_0} dt_0 - e^{a(x-ct)} \int_0^t e^{act_0} dt_0 \right] \\ &= \frac{1}{2ac} \left[e^{a(x+ct)} \left(\frac{1 - e^{-act}}{ac} \right) - e^{a(x-ct)} \left(\frac{e^{act} - 1}{ac} \right) \right] \\ &= \frac{1}{2a^2c^2} [e^{a(x+ct)} - e^{ax} - e^{ax} + e^{a(x-ct)}] \\ &= \frac{e^{ax}}{2a^2c^2} (e^{act} + e^{-act} - 2) \\ &= \frac{e^{ax}}{2a^2c^2} (2 \cosh act - 2) \end{aligned}$$

Therefore,

$$u(x, t) = \frac{e^{ax}}{a^2c^2} (\cosh act - 1).$$

Solution by Duhamel's Principle

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= e^{ax}, & -\infty < x < \infty, t > 0 \\u(x, 0) &= 0 & u_t(x, 0) = 0\end{aligned}$$

According to Duhamel's principle, the solution to the inhomogeneous wave equation is

$$u(x, t) = \int_0^t U(x, t-s; s) ds,$$

where $U = U(x, t; s)$ is the solution to the associated homogeneous equation with a particular choice for the initial conditions.

$$\begin{aligned}U_{tt} - c^2 U_{xx} &= 0, & -\infty < x < \infty, t > 0 \\U(x, 0; s) &= 0 & U_t(x, 0; s) = e^{ax}\end{aligned}$$

The solution for U is given by d'Alembert's formula in section 2.1 on page 36.

$$\begin{aligned}U(x, t; s) &= \frac{1}{2c} \int_{x-ct}^{x+ct} e^{ar} dr \\&= \frac{1}{2c} \frac{e^{ar}}{a} \Big|_{x-ct}^{x+ct} \\&= \frac{1}{2ac} [e^{a(x+ct)} - e^{a(x-ct)}] \\&= \frac{e^{ax}}{2ac} (e^{act} - e^{-act}) \\&= \frac{e^{ax}}{ac} \sinh act\end{aligned}$$

The solution to the inhomogeneous wave equation is then

$$\begin{aligned}u(x, t) &= \int_0^t \frac{e^{ax}}{ac} \sinh[ac(t-s)] ds \\&= \frac{e^{ax}}{ac} \int_0^t \{-\sinh[ac(s-t)]\} ds \\&= \frac{e^{ax}}{ac} \left(\frac{\cosh act - 1}{ac} \right) \\&= \frac{e^{ax}}{ac} (\cosh act - 1).\end{aligned}$$

We can check that the Duhamel solution satisfies the wave equation. Use the Leibnitz rule to differentiate the integrals.

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} \int_0^t U(x, t-s; s) ds \right] - c^2 \frac{\partial^2}{\partial x^2} \int_0^t U(x, t-s; s) ds \\&= \frac{\partial}{\partial t} \left[\int_0^t \frac{\partial}{\partial t} U(x, t-s; s) ds + \underbrace{U(x, 0; t)}_{=0} \cdot 1 - U(x, t; 0) \cdot 0 \right] - c^2 \int_0^t U_{xx}(x, t-s; s) ds \\&= \int_0^t \frac{\partial^2}{\partial t^2} U(x, t-s; s) ds + U_t(x, 0; t) \cdot 1 - U_t(x, t; 0) \cdot 0 - c^2 \int_0^t U_{xx}(x, t-s; s) ds \\&= \int_0^t \underbrace{[U_{tt}(x, t-s; s) - c^2 U_{xx}(x, t-s; s)]}_{=0} ds + U_t(x, 0; t) = e^{ax}\end{aligned}$$