Exercise 3

Solve $u_{tt} = c^2 u_{xx} + \cos x, \quad u(x,0) = \sin x, \quad u_t(x,0) = 1 + x.$

Solution

Solution by Operator Factorization

Bring $c^2 u_{xx}$ to the other side.

$$u_{tt} - c^2 u_{xx} = \cos x$$

Write the left side as an operator acting on u.

$$(\partial_t^2 - c^2 \partial_x^2)u = \cos x$$

The operator is a difference of squares, so it can be factored.

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = \cos x$$

Let

$$v = (\partial_t - c\partial_x)u$$

so that the PDE becomes

$$(\partial_t + c\partial_x)v = \cos x.$$

The second-order PDE we started with has thus been reduced to the following system of first-order PDEs that can be solved with the method of characteristics.

$$u_t - cu_x = v \tag{1}$$

$$v_t + cv_x = \cos x \tag{2}$$

We will solve the second one for v first, and once that is known, the first equation for u will be solved. For a function of two variables $\phi = \phi(x,t)$, its differential is defined as

$$d\phi = \frac{\partial \phi}{\partial t}\, dt + \frac{\partial \phi}{\partial x}\, dx.$$

If we divide both sides by dt, then we get the relationship between the ordinary derivative of ϕ and its partial derivatives.

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{dx}{dt} \tag{3}$$

Comparing this with equation (2), we see that along the curves in the xt-plane that satisfy

$$\frac{dx}{dt} = c, \tag{4}$$

the PDE for $v(x,t)$ reduces to an ODE.

$$\frac{dv}{dt} = \cos x \tag{5}$$

Because c is a constant, equation (4) can be solved by integrating both sides with respect to t.

$$x = ct + \xi, \tag{6}$$

where ξ is a characteristic coordinate. Substitute this expression for x into equation (5) to obtain an ODE that only involves t (ξ is regarded as a constant).

$$\frac{dv}{dt} = \cos(ct + \xi)$$

Integrate both sides with respect to t.

$$v(\xi, t) = \frac{1}{c}\sin(ct + \xi) + f(\xi),$$

where f is an arbitrary function of the characteristic coordinate ξ. In order to write v in terms of x and t, solve equation (6) for ξ.

$$x = ct + \xi \quad \rightarrow \quad \xi = x - ct$$

Hence,

$$v(x, t) = \frac{1}{c}\sin x + f(x - ct).$$

As a result, equation (1) becomes

$$u_t - cu_x = \frac{1}{c}\sin x + f(x - ct).$$

Comparing this equation with equation (3), we see that along the curves in the xt-plane that satisfy

$$\frac{dx}{dt} = -c, \tag{7}$$

the PDE for $u(x, t)$ reduces to an ODE.

$$\frac{du}{dt} = \frac{1}{c}\sin x + f(x - ct) \tag{8}$$

Because c is a constant, equation (7) can be solved by integrating both sides with respect to t.

$$x = -ct + \eta, \tag{9}$$

where η is another characteristic coordinate. Substitute this expression for x into equation (8) to obtain an ODE that only involves t (η is regarded as a constant).

$$\frac{du}{dt} = \frac{1}{c}\sin(-ct + \eta) + f(-ct + \eta - ct)$$

$$\frac{du}{dt} = \frac{1}{c}\sin(-ct + \eta) + f(\eta - 2ct)$$

Integrate both sides with respect to t.

$$u(\eta, t) = \frac{1}{c^2}\cos(-ct + \eta) + \int^t f(\eta - 2cs)\,ds + g(\eta),$$

where g is an arbitrary function of the characteristic coordinate η. The integral of an arbitrary function is another arbitrary function.

$$u(\eta, t) = \frac{1}{c^2}\cos(-ct + \eta) + F(\eta - 2ct) + g(\eta),$$

In order to write u in terms of x and t, solve equation (9) for η.

$$x = -ct + \eta \quad \rightarrow \quad \eta = x + ct$$

Hence,

$$\begin{aligned} u(x,t) &= \frac{1}{c^2}\cos(-ct+\eta) + F(x+ct-2ct) + g(x+ct) \\ &= \frac{1}{c^2}\cos x + F(x-ct) + g(x+ct). \end{aligned}$$

This is the general solution to $u_{tt} = c^2 u_{xx} + \cos x$. If we apply the two initial conditions, we can determine F and g. Before doing so, take a derivative of the solution with respect to t.

$$u_t(x,t) = -cF'(x-ct) + cg'(x+ct)$$

From the initial conditions we obtain the following system of equations.

$$\begin{aligned} u(x,0) &= \frac{1}{c^2}\cos x + F(x) + g(x) = \sin x \\ u_t(x,0) &= -cF'(x) + cg'(x) = 1 + x \end{aligned}$$

Even though this system is in terms of x, it's really in terms of w, where w is any expression we choose.

$$\begin{aligned} \frac{1}{c^2}\cos w + F(w) + g(w) &= \sin w \\ -cF'(w) + cg'(w) &= 1 + w \end{aligned}$$

Differentiating both sides of the first equation with respect to w, we get

$$-\frac{1}{c^2}\sin w + F'(w) + g'(w) = \cos w \quad \rightarrow \quad g'(w) = \cos w + \frac{1}{c^2}\sin w - F'(w).$$

Plug this expression for $g'(w)$ into the second equation.

$$-cF'(w) + c\left[\cos w + \frac{1}{c^2}\sin w - F'(w)\right] = 1 + w \quad \rightarrow \quad F'(w) = \frac{1}{2c}\left(c\cos w + \frac{1}{c}\sin w - w - 1\right)$$

Solve for $F(w)$ and obtain an expression for $F(x-ct)$.

$$\begin{aligned} F(w) &= \frac{1}{2c}\left(c\sin w - \frac{1}{c}\cos w - \frac{w^2}{2} - w\right) + C_1 \\ \Rightarrow \quad F(x-ct) &= \frac{1}{2c}\left[c\sin(x-ct) - \frac{1}{c}\cos(x-ct) - \frac{(x-ct)^2}{2} - (x-ct)\right] + C_1 \end{aligned}$$

Use the first equation to solve for $g(w)$ and obtain an expression for $g(x+ct)$.

$$\begin{aligned} g(w) &= \sin w - \frac{1}{c^2}\cos w - F(w) \\ &= \sin w - \frac{1}{c^2}\cos w - \frac{1}{2c}\left(c\sin w - \frac{1}{c}\cos w - \frac{w^2}{2} - w\right) - C_1 \end{aligned}$$

$$g(w) = \frac{1}{2c}\left(c\sin w - \frac{1}{c}\cos w + \frac{w^2}{2} + w\right) - C_1$$

$$\Rightarrow \quad g(x+ct) = \frac{1}{2c}\left[c\sin(x+ct) - \frac{1}{c}\cos(x+ct) + \frac{(x+ct)^2}{2} + (x+ct)\right] - C_1$$

The general solution for $u(x,t)$ becomes

$$\begin{aligned}
u(x,t) &= \frac{1}{c^2}\cos x + F(x-ct) + g(x+ct) \\
&= \frac{1}{c^2}\cos x + \frac{1}{2c}\left[c\sin(x-ct) - \frac{1}{c}\cos(x-ct) - \frac{(x-ct)^2}{2} - (x-ct)\right] + \cancel{C_1} \\
&\qquad + \frac{1}{2c}\left[c\sin(x+ct) - \frac{1}{c}\cos(x+ct) + \frac{(x+ct)^2}{2} + (x+ct)\right] - \cancel{C_1} \\
&= \frac{1}{c^2}\cos x + \frac{1}{2}[\sin(x-ct) + \sin(x+ct)] - \frac{1}{2c^2}[\cos(x-ct) + \cos(x+ct)] \\
&\qquad + \frac{1}{2c}\left[\frac{(x+ct)^2}{2} - \frac{(x-ct)^2}{2} + (x+ct) - (x-ct)\right] \\
&= \frac{1}{c^2}\cos x + \frac{1}{2}(\sin x\cos ct - \cancel{\cos x\sin ct} + \sin x\cos ct + \cancel{\cos x\sin ct}) \\
&\qquad - \frac{1}{2c^2}(\cos x\cos ct + \cancel{\sin x\sin ct} + \cos x\cos ct - \cancel{\sin x\sin ct}) + \frac{1}{2c}[2ct(x+1)] \\
&= \frac{1}{c^2}\cos x + \sin x\cos ct - \frac{1}{c^2}\cos x\cos ct + t(x+1).
\end{aligned}$$

Therefore,

$$u(x,t) = \frac{1}{c^2}\cos x(1-\cos ct) + \sin x\cos ct + t(x+1).$$

Solution by the Method of Characteristics

Bring c^2u_{xx} to the left side of the PDE.

$$u_{tt} - c^2u_{xx} = \cos x$$

Comparing this with the general form of a second-order PDE,

$$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G,$$

we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = \cos x$. The characteristic equations for a second-order PDE are given by

$$\frac{dx}{dt} = \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC}),$$

the solutions of which are known as the characteristics. Since $B^2 - 4AC = 4c^2 > 0$, the PDE is hyperbolic, so the solutions to these equations are two real and distinct families of characteristic curves in the xt-plane.

$$\frac{dx}{dt} = \frac{1}{2}(\pm\sqrt{4c^2})$$
$$\frac{dx}{dt} = \frac{1}{2}(\pm 2c)$$
$$\frac{dx}{dt} = c \quad \text{or} \quad \frac{dx}{dt} = -c$$

Integrate both sides of each equation with respect to t.

$$x = ct + C_2 \quad \text{or} \quad x = -ct + C_3$$

Now make the substitutions,

$$\xi = x - ct = C_2$$
$$\eta = x + ct = C_3,$$

so that the PDE takes the simplest form. The aim is to write u_{tt}, u_{xx}, and $\cos x$ in terms of the new variables, ξ and η. Solving these two equations for x and t with elimination gives

$$x = \frac{1}{2}(\eta + \xi)$$
$$t = \frac{1}{2c}(\eta - \xi).$$

Use the chain rule to write the old derivatives in terms of the new variables.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = u_\xi(-c) + u_\eta(c) = c(u_\eta - u_\xi)$$
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi(1) + u_\eta(1) = u_\xi + u_\eta$$

Find the second derivatives by using the chain rule again.

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial}{\partial t}(u_\eta - u_\xi) = c\left(\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\right)(u_\eta - u_\xi) = c\left[\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi}(u_\eta - u_\xi) + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}(u_\eta - u_\xi)\right]$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_\xi + u_\eta) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)(u_\xi + u_\eta) = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}(u_\xi + u_\eta) + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}(u_\xi + u_\eta)$$

Hence,

$$\frac{\partial^2 u}{\partial t^2} = c[(-c)(u_{\xi\eta} - u_{\xi\xi}) + (c)(u_{\eta\eta} - u_{\xi\eta})] = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$
$$\frac{\partial^2 u}{\partial x^2} = (1)(u_{\xi\xi} + u_{\xi\eta}) + (1)(u_{\xi\eta} + u_{\eta\eta}) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Substituting these expressions into the PDE, $u_{tt} - c^2 u_{xx} = \cos x$, we obtain

$$c^2(\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}}) - c^2(\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}}) = \cos\left[\frac{1}{2}(\eta + \xi)\right].$$

Simplify the left side.

$$-4c^2 u_{\xi\eta} = \cos\left[\frac{1}{2}(\eta + \xi)\right]$$

Divide both sides by $-4c^2$.

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = -\frac{1}{4c^2}\cos\left[\frac{1}{2}(\eta + \xi)\right]$$

This is known as the first canonical form of the PDE. Integrate both sides of it partially with respect to η.

$$\int^{\eta} \left.\frac{\partial^2 u}{\partial \xi \partial \eta}\right|_{\eta = s} ds = \int^{\eta} -\frac{1}{4c^2}\cos\left[\frac{1}{2}(s + \xi)\right] ds + f(\xi),$$

where f is an arbitrary function of ξ.

$$\frac{\partial u}{\partial \xi} = -\frac{1}{4c^2} \cdot 2\sin\left[\frac{1}{2}(s + \xi)\right]\Bigg|^{\eta} + f(\xi)$$
$$\frac{\partial u}{\partial \xi} = -\frac{1}{2c^2}\sin\left[\frac{1}{2}(\eta + \xi)\right] + f(\xi)$$

Now integrate both sides partially with respect to ξ.

$$\int^{\xi} \left.\frac{\partial u}{\partial \xi}\right|_{\xi = s} ds = \int^{\xi} \left\{-\frac{1}{2c^2}\sin\left[\frac{1}{2}(\eta + s)\right] + f(s)\right\} ds + g(\eta),$$

where g is an arbitrary function of η.

$$u(\xi, \eta) = \left\{\frac{1}{2c^2} \cdot 2\cos\left[\frac{1}{2}(\eta + s)\right] + F(s)\right\}\Bigg|^{\xi} + g(\eta)$$
$$u(\xi, \eta) = \frac{1}{c^2}\cos\left[\frac{1}{2}(\eta + \xi)\right] + F(\xi) + g(\eta)$$

Since u has been solved for, change back to the original variables, x and t, by substituting the expressions for ξ and η.

$$u(x, t) = \frac{1}{c^2}\cos x + F(x - ct) + g(x + ct)$$

This is the general solution to $u_{tt} = c^2 u_{xx} + \cos x$. If we apply the two initial conditions, we can determine F and g. Before doing so, take a derivative of the solution with respect to t.

$$u_t(x, t) = -cF'(x - ct) + cg'(x + ct)$$

From the initial conditions we obtain the following system of equations.

$$\begin{aligned} u(x,0) &= \frac{1}{c^2}\cos x + F(x) + g(x) = \sin x \\ u_t(x,0) &= -cF'(x) + cg'(x) = 1 + x \end{aligned}$$

Even though this system is in terms of x, it's really in terms of w, where w is any expression we choose.

$$\begin{aligned} \frac{1}{c^2}\cos w + F(w) + g(w) &= \sin w \\ -cF'(w) + cg'(w) &= 1 + w \end{aligned}$$

Differentiating both sides of the first equation with respect to w, we get

$$-\frac{1}{c^2}\sin w + F'(w) + g'(w) = \cos w \quad \rightarrow \quad g'(w) = \cos w + \frac{1}{c^2}\sin w - F'(w).$$

Plug this expression for $g'(w)$ into the second equation.

$$-cF'(w) + c\left[\cos w + \frac{1}{c^2}\sin w - F'(w)\right] = 1 + w \quad \rightarrow \quad F'(w) = \frac{1}{2c}\left(c\cos w + \frac{1}{c}\sin w - w - 1\right)$$

Solve for $F(w)$ and obtain an expression for $F(x-ct)$.

$$\begin{aligned} F(w) &= \frac{1}{2c}\left(c\sin w - \frac{1}{c}\cos w - \frac{w^2}{2} - w\right) + C_1 \\ &\Rightarrow \quad F(x-ct) = \frac{1}{2c}\left[c\sin(x-ct) - \frac{1}{c}\cos(x-ct) - \frac{(x-ct)^2}{2} - (x-ct)\right] + C_1 \end{aligned}$$

Use the first equation to solve for $g(w)$ and obtain an expression for $g(x+ct)$.

$$\begin{aligned} g(w) &= \sin w - \frac{1}{c^2}\cos w - F(w) \\ &= \sin w - \frac{1}{c^2}\cos w - \frac{1}{2c}\left(c\sin w - \frac{1}{c}\cos w - \frac{w^2}{2} - w\right) - C_1 \end{aligned}$$

$$\begin{aligned} g(w) &= \frac{1}{2c}\left(c\sin w - \frac{1}{c}\cos w + \frac{w^2}{2} + w\right) - C_1 \\ &\Rightarrow \quad g(x+ct) = \frac{1}{2c}\left[c\sin(x+ct) - \frac{1}{c}\cos(x+ct) + \frac{(x+ct)^2}{2} + (x+ct)\right] - C_1 \end{aligned}$$

The general solution for $u(x,t)$ becomes

$$\begin{aligned}
u(x,t) &= \frac{1}{c^2}\cos x + F(x-ct) + g(x+ct) \\
&= \frac{1}{c^2}\cos x + \frac{1}{2c}\left[c\sin(x-ct) - \frac{1}{c}\cos(x-ct) - \frac{(x-ct)^2}{2} - (x-ct)\right] + \cancel{C_1} \\
&\qquad + \frac{1}{2c}\left[c\sin(x+ct) - \frac{1}{c}\cos(x+ct) + \frac{(x+ct)^2}{2} + (x+ct)\right] - \cancel{C_1} \\
&= \frac{1}{c^2}\cos x + \frac{1}{2}[\sin(x-ct) + \sin(x+ct)] - \frac{1}{2c^2}[\cos(x-ct) + \cos(x+ct)] \\
&\qquad + \frac{1}{2c}\left[\frac{(x+ct)^2}{2} - \frac{(x-ct)^2}{2} + (x+ct) - (x-ct)\right] \\
&= \frac{1}{c^2}\cos x + \frac{1}{2}(\sin x\cos ct - \cancel{\cos x\sin ct} + \sin x\cos ct + \cancel{\cos x\sin ct}) \\
&\qquad - \frac{1}{2c^2}(\cos x\cos ct + \cancel{\sin x\sin ct} + \cos x\cos ct - \cancel{\sin x\sin ct}) + \frac{1}{2c}[2ct(x+1)] \\
&= \frac{1}{c^2}\cos x + \sin x\cos ct - \frac{1}{c^2}\cos x\cos ct + t(x+1).
\end{aligned}$$

Therefore,

$$u(x,t) = \frac{1}{c^2}\cos x(1-\cos ct) + \sin x\cos ct + t(x+1).$$

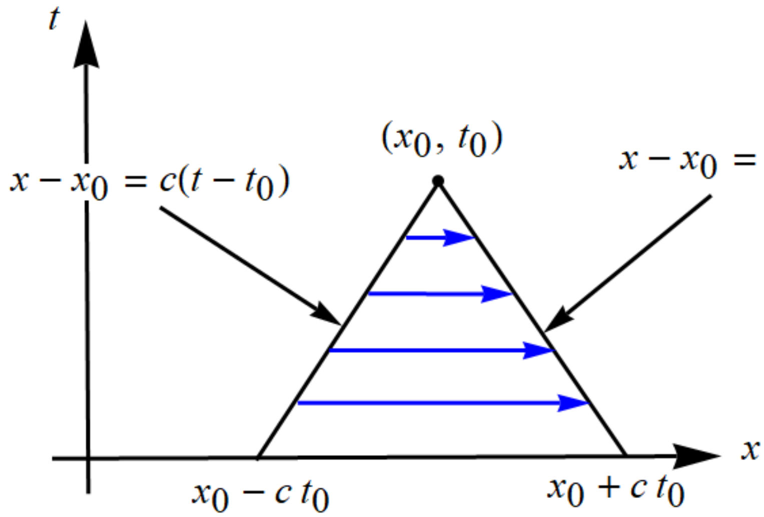
Solution by Green's Theorem

$$u_{tt} - c^2 u_{xx} = \cos x, \quad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = \sin x \qquad u_t(x,0) = 1 + x$$

The characteristics were found to be straight lines, $\xi = x - ct$ and $\eta = x + ct$, with slopes $\pm c$. Suppose (x_0, t_0) is the point in the xt-plane we want to evaluate u at. The equations of the lines going through this point are

$$x - x_0 = c(t - t_0)$$
$$x - x_0 = -c(t - t_0).$$

Integrate both sides of the inhomogeneous wave equation over the triangular domain D enclosed by these lines (from left to right as indicated below).



Write the double integral explicitly on the right side.

$$\iint_D (u_{tt} - c^2 u_{xx})\, dA = \int_0^{t_0} \int_{x_0 + c(t-t_0)}^{x_0 - c(t-t_0)} \cos x\, dx\, dt$$

Rewrite the left side.

$$-\iint_D \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = \int_0^{t_0} \int_{x_0 + c(t-t_0)}^{x_0 - c(t-t_0)} \cos x\, dx\, dt$$

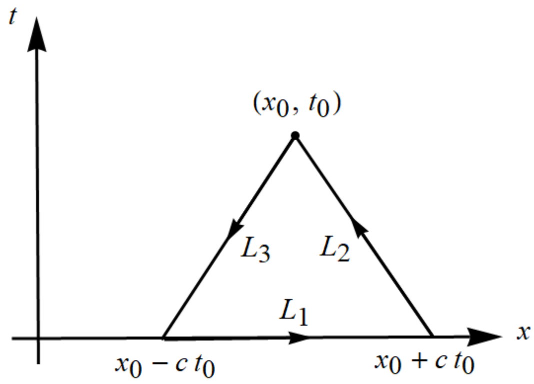
Multiply both sides by -1.

$$\iint_D \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = -\int_0^{t_0} \int_{x_0 + c(t-t_0)}^{x_0 - c(t-t_0)} \cos x\, dx\, dt$$

Apply Green's theorem (essentially the divergence theorem in two dimensions) to the double integral on the left to turn it into a counterclockwise line integral around the triangle's boundary bdy D.

$$\oint_{\text{bdy } D} (u_t\, dx + c^2 u_x\, dt) = -\int_0^{t_0} \int_{x_0 + c(t-t_0)}^{x_0 - c(t-t_0)} \cos x\, dx\, dt$$

Let L_1, L_2, and L_3 represent the legs of the triangle.



The line integral is the sum of three integrals, one over each leg.

$$\int_{L_1}(u_t\,dx + c^2u_x\,dt) + \int_{L_2}(u_t\,dx + c^2u_x\,dt) + \int_{L_3}(u_t\,dx + c^2u_x\,dt) = -\int_0^{t_0}\int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} \cos x\,dx\,dt$$

On L_1	On L_2	On L_3
$t = 0$	$x - x_0 = -c(t - t_0)$	$x - x_0 = c(t - t_0)$
$dt = 0$	$dx = -c\,dt$	$dx = c\,dt$

Replace the differentials in the integrals over L_2 and L_3.

$$\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\,dx + \int_{L_2}\left[u_t(-c\,dt) + c^2u_x\left(-\frac{dx}{c}\right)\right] + \int_{L_3}\left[u_t(c\,dt) + c^2u_x\left(\frac{dx}{c}\right)\right] = -\int_0^{t_0}\int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} \cos x\,dx\,dt$$

In this exercise $u_t(x,0) = 1 + x$.

$$\int_{x_0-ct_0}^{x_0+ct_0}(1+x)\,dx - c\int_{L_2}\left(\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx\right) + c\int_{L_3}\left(\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx\right) = -\int_0^{t_0}\int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} \cos x\,dx\,dt$$

The second and third integrands on the left side are how the differential of $u = u(x,t)$ is defined.

$$\int_{x_0-ct_0}^{x_0+ct_0}(1+x)\,dx - c\int_{L_2} du + c\int_{L_3} du = -\int_0^{t_0}\int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} \cos x\,dx\,dt$$

Evaluate the second and third integrals on the left side.

$$\int_{x_0-ct_0}^{x_0+ct_0}(1+x)\,dx - c[u(x_0,t_0)-u(x_0+ct_0,0)] + c[u(x_0-ct_0,0)-u(x_0,t_0)] = -\int_0^{t_0}\int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} \cos x\,dx\,dt$$

In this exercise $u(x,0) = \sin x$, so $u(x_0+ct_0,0) = \sin(x_0+ct_0)$ and $u(x_0-ct_0,0) = \sin(x_0-ct_0)$.

$$\int_{x_0-ct_0}^{x_0+ct_0}(1+x)\,dx - 2cu(x_0,t_0) + c[\sin(x_0+ct_0) + \sin(x_0-ct_0)] = -\int_0^{t_0}\int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} \cos x\,dx\,dt$$

Solve this equation for $2cu(x_0,t_0)$.

$$2cu(x_0,t_0) = c[\sin(x_0+ct_0) + \sin(x_0-ct_0)] + \int_{x_0-ct_0}^{x_0+ct_0}(1+x)\,dx + \int_0^{t_0}\int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} \cos x\,dx\,dt$$

Divide both sides by $2c$.

$$u(x_0,t_0)=\frac{1}{2}[\sin(x_0+ct_0)+\sin(x_0-ct_0)]+\frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0}(1+x)\,dx+\frac{1}{2c}\int_0^{t_0}\int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)}\cos x\,dx\,dt$$

Finally, switch the roles of x and t with those of x_0 and t_0, respectively.

$$u(x,t)=\frac{1}{2}[\sin(x+ct)+\sin(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}(1+x_0)\,dx_0+\frac{1}{2c}\int_0^{t}\int_{x+c(t_0-t)}^{x-c(t_0-t)}\cos x_0\,dx_0\,dt_0$$

Proceed to evaluate the last integrals.

$$\begin{aligned}
u(x,t)&=\frac{1}{2}[\sin(x+ct)+\sin(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}(1+x_0)\,dx_0+\frac{1}{2c}\int_0^{t}\int_{x-c(t-t_0)}^{x+c(t-t_0)}\cos x_0\,dx_0\,dt_0\\
&=\frac{1}{2}(2\cos ct\sin x)+\frac{1}{2c}\left(x_0+\frac{x_0^2}{2}\right)\bigg|_{x-ct}^{x+ct}+\frac{1}{2c}\int_0^t\sin x_0\bigg|_{x-c(t-t_0)}^{x+c(t-t_0)}\,dt_0\\
&=\cos ct\sin x+\frac{1}{2c}(2ct+2ctx)+\frac{1}{2c}\int_0^t\{\sin[x+c(t-t_0)]-\sin[x-c(t-t_0)]\}\,dt_0\\
&=\cos ct\sin x+t+tx+\frac{1}{2c}\left(\frac{\cos x-\cos(x+ct)}{c}-\frac{\cos(x-ct)-\cos x}{c}\right)\\
&=\cos ct\sin x+t(x+1)-\frac{1}{2c^2}[\cos(x+ct)+\cos(x-ct)]+\frac{1}{c^2}\cos x\\
&=\cos ct\sin x+t(x+1)-\frac{1}{2c^2}(2\cos ct\cos x)+\frac{1}{c^2}\cos x\\
&=\cos ct\sin x+t(x+1)-\frac{1}{c^2}\cos ct\cos x+\frac{1}{c^2}\cos x
\end{aligned}$$

Therefore,

$$u(x,t)=\frac{1}{c^2}\cos x(1-\cos ct)+\sin x\cos ct+t(x+1).$$

Solution by Duhamel's Principle

$$\begin{gathered} u_{tt} - c^2 u_{xx} = \cos x, \quad -\infty < x < \infty,\ t > 0 \\ u(x,0) = \sin x \qquad u_t(x,0) = 1 + x \end{gathered}$$

Use the fact that the PDE is linear to split up the problem. Let $u(x,t) = v(x,t) + w(x,t)$, where v and w satisfy the following initial value problems.

$$\begin{aligned} &v_{tt} - c^2 v_{xx} = 0 & &w_{tt} - c^2 w_{xx} = \cos x \\ &v(x,0) = \sin x \quad v_t(x,0) = 1 + x & &w(x,0) = 0 \quad w_t(x,0) = 0 \end{aligned}$$

The solution for v is given by d'Alembert's formula in section 2.1 on page 36.

$$\begin{aligned} v(x,t) &= \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} (1 + x_0)\, dx_0 \\ &= \cos ct \sin x + t(x+1) \end{aligned}$$

According to Duhamel's principle, the solution to the inhomogeneous wave equation is

$$w(x,t) = \int_0^t W(x, t-s; s)\, ds,$$

where $W = W(x,t;s)$ is the solution to the associated homogeneous equation with a particular choice for the initial conditions.

$$\begin{gathered} W_{tt} - c^2 W_{xx} = 0, \quad -\infty < x < \infty,\ t > 0 \\ W(x,0;s) = 0 \qquad W_t(x,0;s) = \cos x \end{gathered}$$

The solution for W is given by d'Alembert's formula.

$$\begin{aligned} W(x,t;s) &= \frac{1}{2c}\int_{x-ct}^{x+ct} \cos r\, dr \\ &= \left.\frac{1}{2c}\sin r\right|_{x-ct}^{x+ct} \\ &= \frac{1}{2c}[\sin(x+ct) - \sin(x-ct)] \\ &= \frac{1}{2c}[2\cos x \sin ct] \\ &= \frac{1}{c}\cos x \sin ct \end{aligned}$$

The solution to the inhomogeneous wave equation is then

$$\begin{aligned} w(x,t) &= \int_0^t \frac{1}{c}\cos x \sin[c(t-s)]\, ds \\ &= \frac{1}{c}\cos x \int_0^t \{-\sin[c(s-t)]\}\, ds \\ &= \frac{1}{c}\cos x \left(\frac{1 - \cos ct}{c}\right) \\ &= \frac{1}{c^2}\cos x (1 - \cos ct). \end{aligned}$$

Therefore,

$$u(x,t) = \frac{1}{c^2}\cos x(1-\cos ct) + \sin x\cos ct + t(x+1).$$

We can check that the Duhamel solution satisfies the wave equation. Use the Leibnitz rule to differentiate the integrals.

$$\begin{aligned}
w_{tt} - c^2 w_{xx} &= \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\int_0^t W(x,t-s;s)\,ds\right] - c^2\frac{\partial^2}{\partial x^2}\int_0^t W(x,t-s;s)\,ds \\
&= \frac{\partial}{\partial t}\left[\int_0^t \frac{\partial}{\partial t}W(x,t-s;s)\,ds + \underbrace{W(x,0;t)}_{=0}\cdot 1 - W(x,t;0)\cdot 0\right] - c^2\int_0^t W_{xx}(x,t-s;s)\,ds \\
&= \int_0^t \frac{\partial^2}{\partial t^2}W(x,t-s;s)\,ds + W_t(x,0;t)\cdot 1 - W_t(x,t;0)\cdot 0 - c^2\int_0^t W_{xx}(x,t-s;s)\,ds \\
&= \int_0^t [\underbrace{W_{tt}(x,t-s;s) - c^2 W_{xx}(x,t-s;s)}_{=0}]\,ds + W_t(x,0;t) = \cos x
\end{aligned}$$