Exercise 4

Show that the solution of the inhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + f, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x),$$

is the sum of three terms, one each for f, ϕ, and ψ.

Solution

Solution by Operator Factorization

Bring $c^2 u_{xx}$ to the other side.

$$u_{tt} - c^2 u_{xx} = f(x, t)$$

Write the left side as an operator acting on u.

$$(\partial_t^2 - c^2 \partial_x^2) u = f(x, t)$$

The operator is a difference of squares, so it can be factored.

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x) u = f(x, t)$$

Let

$$v = (\partial_t - c\partial_x) u$$

so that the PDE becomes

$$(\partial_t + c\partial_x) v = f(x, t).$$

The second-order PDE we started with has thus been reduced to the following system of first-order PDEs that can be solved with the method of characteristics.

$$u_t - cu_x = v \tag{1}$$
$$v_t + cv_x = f(x, t) \tag{2}$$

We will solve the second one for v first, and once that is known, the first equation for u will be solved. For a function of two variables $z = z(x, t)$, its differential is defined as

$$dz = \frac{\partial z}{\partial t}\, dt + \frac{\partial z}{\partial x}\, dx.$$

If we divide both sides by dt, then we get the relationship between the ordinary derivative of ϕ and its partial derivatives.

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x}\frac{dx}{dt} \tag{3}$$

Comparing this with equation (2), we see that along the curves in the xt-plane that satisfy

$$\frac{dx}{dt} = c, \tag{4}$$

the PDE for $v(x, t)$ reduces to an ODE.

$$\frac{dv}{dt} = f(x, t) \tag{5}$$

Because c is a constant, equation (4) can be solved by integrating both sides with respect to t.

$$x = ct + \xi, \tag{6}$$

where ξ is a characteristic coordinate. Substitute this expression for x into equation (5) to obtain an ODE that only involves t (ξ is regarded as a constant).

$$\frac{dv}{dt} = f(ct + \xi, t)$$

Integrate both sides with respect to t.

$$v(\xi, t) = \int_0^t f(cs + \xi, s)\, ds + g(\xi),$$

where g is an arbitrary function of the characteristic coordinate ξ. Note that because g is present, the lower limit of integration is arbitrary and has been set equal to 0. In order to write v in terms of x and t, solve equation (6) for ξ.

$$x = ct + \xi \quad \rightarrow \quad \xi = x - ct$$

Hence,

$$v(x, t) = \int_0^t f(cs + x - ct, s)\, ds + g(x - ct).$$

As a result, equation (1) becomes

$$u_t - cu_x = \int_0^t f(cs + x - ct, s)\, ds + g(x - ct).$$

Comparing this equation with equation (3), we see that along the curves in the xt-plane that satisfy

$$\frac{dx}{dt} = -c, \tag{7}$$

the PDE for $u(x, t)$ reduces to an ODE.

$$\frac{du}{dt} = \int_0^t f(cs + x - ct, s)\, ds + g(x - ct) \tag{8}$$

Because c is a constant, equation (7) can be solved by integrating both sides with respect to t.

$$x = -ct + \eta, \tag{9}$$

where η is another characteristic coordinate. Substitute this expression for x into equation (8) to obtain an ODE that only involves t (η is regarded as a constant).

$$\frac{du}{dt} = \int_0^t f(cs - ct + \eta - ct, s)\, ds + g(-ct + \eta - ct)$$

$$\frac{du}{dt} = \int_0^t f(cs + \eta - 2ct, s)\, ds + g(\eta - 2ct)$$

Integrate both sides with respect to t.

$$u(\eta, t) = \int_0^t \left[\int_0^r f(cs + \eta - 2cr, s)\, ds\right] dr + \int^t g(\eta - 2cr)\, dr + h(\eta)$$

where h is an arbitrary function of the characteristic coordinate η. Again, the lower limit of integration is arbitrary and has been set equal to 0. The integral of an arbitrary function is another arbitrary function.

$$u(\eta, t) = \int_0^t \int_0^r f(cs + \eta - 2cr, s)\, ds\, dr + G(\eta - 2ct) + h(\eta)$$

In order to write u in terms of x and t, solve equation (9) for η.

$$x = -ct + \eta \quad \rightarrow \quad \eta = x + ct$$

Hence,

$$\begin{aligned} u(x, t) &= \int_0^t \int_0^r f(cs + x + ct - 2cr, s)\, ds\, dr + G(x + ct - 2ct) + h(x + ct) \\ &= \int_0^t \int_0^r f(x + ct + cs - 2cr, s)\, ds\, dr + G(x - ct) + h(x + ct). \end{aligned}$$

In order to simplify the double integral we will switch the order of integration. At the moment, the inner integral is in ds, and s is present in both of f's arguments. r, on the other hand, is only in the first argument, so we can simplify the integrand if we make the inner integral in dr.

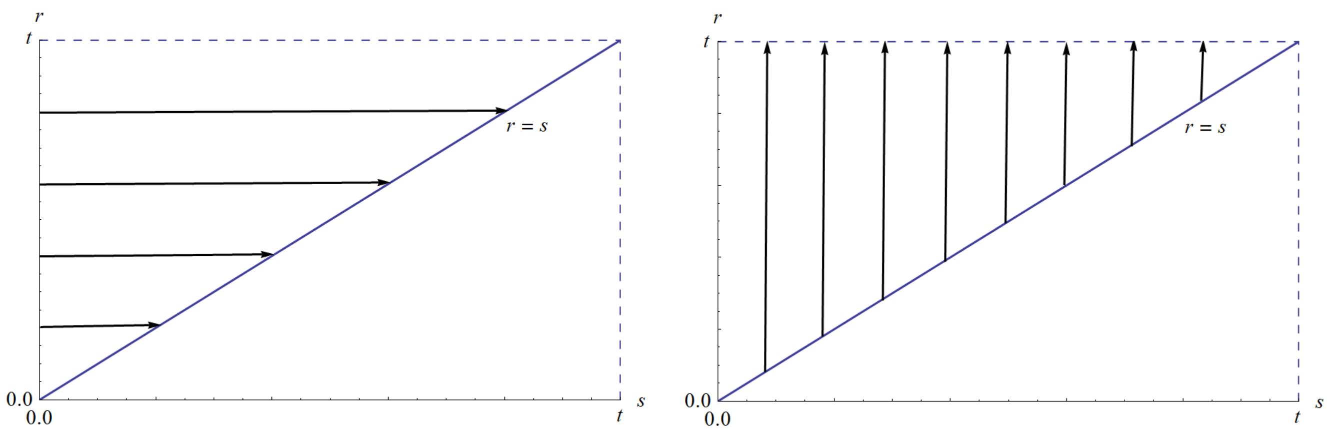


Figure 1: The current mode of integration in the sr-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$u(x, t) = \int_0^t \int_s^t f(x + ct + cs - 2cr, s)\, dr\, ds + G(x - ct) + h(x + ct)$$

Now the following substitution can be made in the integral.

$$\begin{aligned} y &= x + ct + cs - 2cr \\ dy &= -2c\, dr \quad \rightarrow \quad -\frac{1}{2c}\, dy = dr \end{aligned}$$

The formula for u becomes

$$u(x, t) = \int_0^t \int_{x+ct-cs}^{x-ct+cs} f(y, s) \left(-\frac{1}{2c}\, dy\right) ds + G(x - ct) + h(x + ct).$$

Bring the constant out in front and use the minus sign to switch the limits of integration.

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)\,dy\,ds + G(x-ct) + h(x+ct)$$

Therefore,

$$u(x,t) = G(x-ct) + h(x+ct) + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.$$

This is the general solution to $u_{tt} = c^2u_{xx} + f$. If we apply the two initial conditions, we can determine G and h. Before doing so, take a derivative of the solution with respect to t.

$$\begin{aligned}
u_t(x,t) &= -cG'(x-ct) + ch'(x+ct) + \frac{1}{2c}\frac{\partial}{\partial t}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\
&= -cG'(x-ct) + ch'(x+ct) + \frac{1}{2c}\int_0^t \left[\frac{\partial}{\partial t}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right] ds + \underbrace{\int_x^x f(y,t)\,dy}_{=0} \\
&= -cG'(x-ct) + ch'(x+ct) + \frac{1}{2c}\int_0^t \left\{\int_{x-c(t-s)}^{x+c(t-s)} \underbrace{\frac{\partial}{\partial t} f(y,s)}_{=0}\,dy + f[x+c(t-s),s]\times(c)\right. \\
&\qquad\qquad \left. - f[x-c(t-s),s]\times(-c)\right\} ds \\
&= -cG'(x-ct) + ch'(x+ct) + \frac{1}{2}\int_0^t \{f[x+c(t-s),s] + f[x-c(t-s),s]\}\,ds
\end{aligned}$$

In differentiating the double integral, I made use of the Leibnitz integration rule which states that if

$$I(t) = \int_{a(t)}^{b(t)} \gamma(x,t)\,dx,$$

then

$$\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial \gamma}{\partial t}\,dx + \gamma[b(t),t]b'(t) - \gamma[a(t),t]a'(t).$$

From the initial conditions we obtain the following system of equations.

$$\begin{aligned}
u(x,0) &= G(x) + h(x) = \phi(x) \\
u_t(x,0) &= -cG'(x) + ch'(x) = \psi(x)
\end{aligned}$$

Even though this system is in terms of x, it's really in terms of w, where w is any expression we choose.

$$\begin{aligned}
G(w) + h(w) &= \phi(w) \\
-cG'(w) + ch'(w) &= \psi(w)
\end{aligned}$$

Differentiating both sides of the first equation with respect to w, we get

$$G'(w) + h'(w) = \phi'(w) \quad \to \quad h'(w) = \phi'(w) - G'(w).$$

Plug this expression for $h'(w)$ into the second equation.

$$-cG'(w) + c[\phi'(w) - G'(w)] = \psi(w) \quad \to \quad -2cG'(w) + c\phi'(w) = \psi(w) \quad \to \quad G'(w) = \frac{1}{2}\phi'(w) - \frac{1}{2c}\psi(w).$$

Solve for $G(w)$ and obtain an expression for $G(x - ct)$.

$$G(w) = \frac{1}{2}\phi(w) - \int^w \frac{1}{2c}\psi(s)\,ds + C_1 \quad \Rightarrow \quad G(x-ct) = \frac{1}{2}\phi(x-ct) - \int^{x-ct} \frac{1}{2c}\psi(s)\,ds + C_1$$

Use the first equation to solve for $h(w)$ and obtain an expression for $h(x + ct)$.

$$\begin{aligned} h(w) &= \phi(w) - G(w) \\ &= \phi(w) - \frac{1}{2}\phi(w) + \int^w \frac{1}{2c}\psi(s)\,ds - C_1 \\ &= \frac{1}{2}\phi(w) + \int^w \frac{1}{2c}\psi(s)\,ds - C_1 \quad \Rightarrow \quad h(x+ct) = \frac{1}{2}\phi(x+ct) + \int^{x+ct} \frac{1}{2c}\psi(s)\,ds - C_1 \end{aligned}$$

The general solution for $u(x, t)$ becomes

$$\begin{aligned} u(x,t) &= G(x-ct) + h(x+ct) + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\ &= \frac{1}{2}\phi(x-ct) - \int^{x-ct} \frac{1}{2c}\psi(s)\,ds + \cancel{C_1} + \frac{1}{2}\phi(x+ct) + \int^{x+ct} \frac{1}{2c}\psi(s)\,ds - \cancel{C_1} \\ &\qquad + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\ &= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \int_{x-ct} \frac{1}{2c}\psi(s)\,ds + \int^{x+ct} \frac{1}{2c}\psi(s)\,ds \\ &\qquad + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\ &= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \int_{x-ct}^{x+ct} \frac{1}{2c}\psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \end{aligned}$$

Therefore, $u(x, t)$ is the sum of three terms—one involving f, one involving ϕ, and one involving ψ.

$$u(x,t) = \underbrace{\frac{1}{2}[\phi(x+ct) + \phi(x-ct)]}_{1} + \underbrace{\frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds}_{2} + \underbrace{\frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds}_{3}$$

Solution by the Method of Characteristics

Bring c^2u_{xx} to the left side of the PDE.

$$u_{tt} - c^2u_{xx} = f(x,t)$$

Comparing this with the general form of a second-order PDE,

$$Au_{tt} + Bu_{xt} + Cu_{xx} + Du_t + Eu_x + Fu = G,$$

we see that $A = 1$, $B = 0$, $C = -c^2$, $D = 0$, $E = 0$, $F = 0$, and $G = f(x,t)$. The characteristic equations for a second-order PDE are given by

$$\frac{dx}{dt} = \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC}),$$

the solutions of which are known as the characteristics. Since $B^2 - 4AC = 4c^2 > 0$, the PDE is hyperbolic, so the solutions to these equations are two real and distinct families of characteristic curves in the xt-plane.

$$\frac{dx}{dt} = \frac{1}{2}(\pm\sqrt{4c^2})$$
$$\frac{dx}{dt} = \frac{1}{2}(\pm 2c)$$
$$\frac{dx}{dt} = c \quad \text{or} \quad \frac{dx}{dt} = -c$$

Integrate both sides of each equation with respect to t.

$$x = ct + C_2 \quad \text{or} \quad x = -ct + C_3$$

Now make the substitutions,

$$\xi = x - ct = C_2$$
$$\eta = x + ct = C_3,$$

so that the PDE takes the simplest form. The aim is to write u_{tt}, u_{xx}, and $f(x,t)$ in terms of the new variables, ξ and η. Solving these two equations for x and t with elimination gives

$$x = \frac{1}{2}(\eta + \xi)$$
$$t = \frac{1}{2c}(\eta - \xi).$$

Use the chain rule to write the old derivatives in terms of the new variables.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = u_\xi(-c) + u_\eta(c) = c(u_\eta - u_\xi)$$
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi(1) + u_\eta(1) = u_\xi + u_\eta$$

Find the second derivatives by using the chain rule again.

$$\frac{\partial^2 u}{\partial t^2} = c\frac{\partial}{\partial t}(u_\eta - u_\xi) = c\left(\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\right)(u_\eta - u_\xi) = c\left[\frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi}(u_\eta - u_\xi) + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}(u_\eta - u_\xi)\right]$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_\xi + u_\eta) = \left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)(u_\xi + u_\eta) = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi}(u_\xi + u_\eta) + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}(u_\xi + u_\eta)$$

Hence,

$$\frac{\partial^2 u}{\partial t^2} = c[(-c)(u_{\xi\eta} - u_{\xi\xi}) + (c)(u_{\eta\eta} - u_{\xi\eta})] = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$
$$\frac{\partial^2 u}{\partial x^2} = (1)(u_{\xi\xi} + u_{\xi\eta}) + (1)(u_{\xi\eta} + u_{\eta\eta}) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Substituting these expressions into the PDE, $u_{tt} - c^2 u_{xx} = f(x,t)$, we obtain

$$c^2(\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}}) - c^2(\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}}) = f\left[\frac{1}{2}(\eta+\xi), \frac{1}{2c}(\eta-\xi)\right].$$

Simplify the left side.

$$-4c^2 u_{\xi\eta} = f\left[\frac{1}{2}(\eta+\xi), \frac{1}{2c}(\eta-\xi)\right]$$

Divide both sides by $-4c^2$.

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = -\frac{1}{4c^2} f\left[\frac{1}{2}(\eta+\xi), \frac{1}{2c}(\eta-\xi)\right]$$

This is known as the first canonical form of the PDE. Integrate both sides of it partially with respect to η.

$$\int^{\eta} \left.\frac{\partial^2 u}{\partial\xi\partial\eta}\right|_{\eta=r} dr = \int_{\xi}^{\eta} -\frac{1}{4c^2} f\left[\frac{1}{2}(r+\xi), \frac{1}{2c}(r-\xi)\right] dr + g(\xi),$$

where g is an arbitrary function of ξ. The lower limit of integration is arbitrary and has been set equal to ξ. In order to simplify the integrand, make the following substitution.

$$s = \frac{1}{2c}(r-\xi) \quad \rightarrow \quad cs = \frac{1}{2}(r-\xi) \quad \rightarrow \quad cs + \xi = \frac{1}{2}(r+\xi)$$
$$ds = \frac{1}{2c}\,dr \quad \rightarrow \quad 2c\,ds = dr$$

The equation becomes

$$\frac{\partial u}{\partial\xi} = \int_0^{\frac{1}{2c}(\eta-\xi)} -\frac{1}{4c^2} f(cs+\xi, s)(2c\,ds) + g(\xi).$$

Bring the constants in front of the integral.

$$\frac{\partial u}{\partial\xi} = -\frac{1}{2c}\int_0^{\frac{1}{2c}(\eta-\xi)} f(cs+\xi, s)\,ds + g(\xi)$$

Now integrate both sides partially with respect to ξ.

$$\int^{\xi} \left.\frac{\partial u}{\partial\xi}\right|_{\xi=p} dp = -\frac{1}{2c}\int_{\eta}^{\xi}\int_0^{\frac{1}{2c}(\eta-p)} f(cs+p, s)\,ds\,dp + \int^{\xi} g(p)\,dp + h(\eta),$$

where h is an arbitrary function of η. The lower limit of integration is arbitrary and has been set equal to η. The integral of an arbitrary function is another arbitrary function.

$$u(\xi,\eta) = -\frac{1}{2c}\int_{\eta}^{\xi}\int_0^{\frac{1}{2c}(\eta-p)} f(cs+p, s)\,ds\,dp + G(\xi) + h(\eta)$$

Use the minus sign to switch the limits of the first integral.

$$u(\xi,\eta) = \frac{1}{2c}\int_{\xi}^{\eta}\int_{0}^{\frac{1}{2c}(\eta-p)} f(cs+p,s)\,ds\,dp + G(\xi) + h(\eta)$$

In order to simplify the double integral we will switch the order of integration. At the moment, the inner integral is in ds, and s is present in both of f's arguments. p, on the other hand, is only in the first argument, so we can simplify the integrand if we make the inner integral in dp.

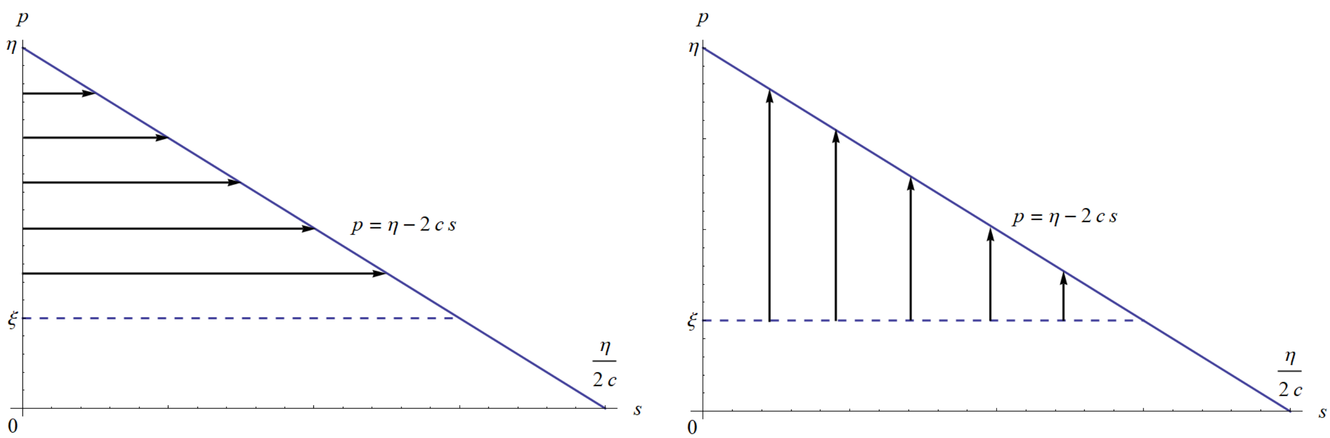


Figure 2: The current mode of integration in the sp-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$u(\xi,\eta) = \frac{1}{2c}\int_{0}^{\frac{1}{2c}(\eta-\xi)}\int_{\xi}^{\eta-2cs} f(cs+p,s)\,dp\,ds + G(\xi) + h(\eta)$$

Now the following substitution can be made.

$$\begin{aligned} y &= cs+p \\ dy &= dp \end{aligned}$$

As a result,

$$\begin{aligned} u(\xi,\eta) &= \frac{1}{2c}\int_{0}^{\frac{1}{2c}(\eta-\xi)}\int_{cs+\xi}^{cs+\eta-2cs} f(y,s)\,dy\,ds + G(\xi) + h(\eta) \\ &= \frac{1}{2c}\int_{0}^{\frac{1}{2c}(\eta-\xi)}\int_{\xi+cs}^{\eta-cs} f(y,s)\,dy\,ds + G(\xi) + h(\eta). \end{aligned}$$

Change back to the original variables now that u is solved for.

$$u(x,t) = \frac{1}{2c}\int_{0}^{t}\int_{x-ct+cs}^{x+ct-cs} f(y,s)\,dy\,ds + G(x-ct) + h(x+ct)$$

Therefore,

$$u(x,t) = G(x-ct) + h(x+ct) + \frac{1}{2c}\int_{0}^{t}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.$$

This is the general solution to $u_{tt} = c^2 u_{xx} + f$. If we apply the two initial conditions, we can determine G and h. Before doing so, take a derivative of the solution with respect to t.

$$\begin{aligned}
u_t(x,t) &= -cG'(x-ct) + ch'(x+ct) + \frac{1}{2c}\frac{\partial}{\partial t}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\
&= -cG'(x-ct) + ch'(x+ct) + \frac{1}{2c}\int_0^t \left[\frac{\partial}{\partial t}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right] ds + \underbrace{\int_x^x f(y,t)\,dy}_{=0} \\
&= -cG'(x-ct) + ch'(x+ct) + \frac{1}{2c}\int_0^t \left\{\int_{x-c(t-s)}^{x+c(t-s)} \underbrace{\frac{\partial}{\partial t} f(y,s)}_{=0}\,dy + f[x+c(t-s),s]\times(c) \right. \\
&\qquad\qquad \left. - f[x-c(t-s),s]\times(-c)\right\} ds \\
&= -cG'(x-ct) + ch'(x+ct) + \frac{1}{2}\int_0^t \{f[x+c(t-s),s] + f[x-c(t-s),s]\}\,ds
\end{aligned}$$

In differentiating the double integral, I made use of the Leibnitz integration rule which states that if

$$I(t) = \int_{a(t)}^{b(t)} \gamma(x,t)\,dx,$$

then

$$\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial \gamma}{\partial t}\,dx + \gamma[b(t),t]b'(t) - \gamma[a(t),t]a'(t).$$

From the initial conditions we obtain the following system of equations.

$$\begin{aligned}
u(x,0) &= G(x) + h(x) = \phi(x) \\
u_t(x,0) &= -cG'(x) + ch'(x) = \psi(x)
\end{aligned}$$

Even though this system is in terms of x, it's really in terms of w, where w is any expression we choose.

$$\begin{aligned}
G(w) + h(w) &= \phi(w) \\
-cG'(w) + ch'(w) &= \psi(w)
\end{aligned}$$

Differentiating both sides of the first equation with respect to w, we get

$$G'(w) + h'(w) = \phi'(w) \quad \to \quad h'(w) = \phi'(w) - G'(w).$$

Plug this expression for $h'(w)$ into the second equation.

$$-cG'(w) + c[\phi'(w) - G'(w)] = \psi(w) \quad \to \quad -2cG'(w) + c\phi'(w) = \psi(w) \quad \to \quad G'(w) = \frac{1}{2}\phi'(w) - \frac{1}{2c}\psi(w).$$

Solve for $G(w)$ and obtain an expression for $G(x-ct)$.

$$G(w) = \frac{1}{2}\phi(w) - \int^w \frac{1}{2c}\psi(s)\,ds + C_1 \quad \Rightarrow \quad G(x-ct) = \frac{1}{2}\phi(x-ct) - \int^{x-ct} \frac{1}{2c}\psi(s)\,ds + C_1$$

Use the first equation to solve for $h(w)$ and obtain an expression for $h(x+ct)$.

$$\begin{aligned}
h(w) &= \phi(w) - G(w) \\
&= \phi(w) - \frac{1}{2}\phi(w) + \int^w \frac{1}{2c}\psi(s)\,ds - C_1 \\
&= \frac{1}{2}\phi(w) + \int^w \frac{1}{2c}\psi(s)\,ds - C_1 \quad \Rightarrow \quad h(x+ct) = \frac{1}{2}\phi(x+ct) + \int^{x+ct} \frac{1}{2c}\psi(s)\,ds - C_1
\end{aligned}$$

The general solution for $u(x,t)$ becomes

$$\begin{aligned}
u(x,t) &= G(x-ct) + h(x+ct) + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\
&= \frac{1}{2}\phi(x-ct) - \int^{x-ct} \frac{1}{2c}\psi(s)\,ds + \cancel{C_1} + \frac{1}{2}\phi(x+ct) + \int^{x+ct} \frac{1}{2c}\psi(s)\,ds - \cancel{C_1} \\
&\qquad + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\
&= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \int_{x-ct} \frac{1}{2c}\psi(s)\,ds + \int^{x+ct} \frac{1}{2c}\psi(s)\,ds \\
&\qquad + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \\
&= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \int_{x-ct}^{x+ct} \frac{1}{2c}\psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.
\end{aligned}$$

Therefore, $u(x,t)$ is the sum of three terms—one involving f, one involving ϕ, and one involving ψ.

$$u(x,t) = \underbrace{\frac{1}{2}[\phi(x+ct) + \phi(x-ct)]}_{1} + \underbrace{\frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds}_{2} + \underbrace{\frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds}_{3}$$

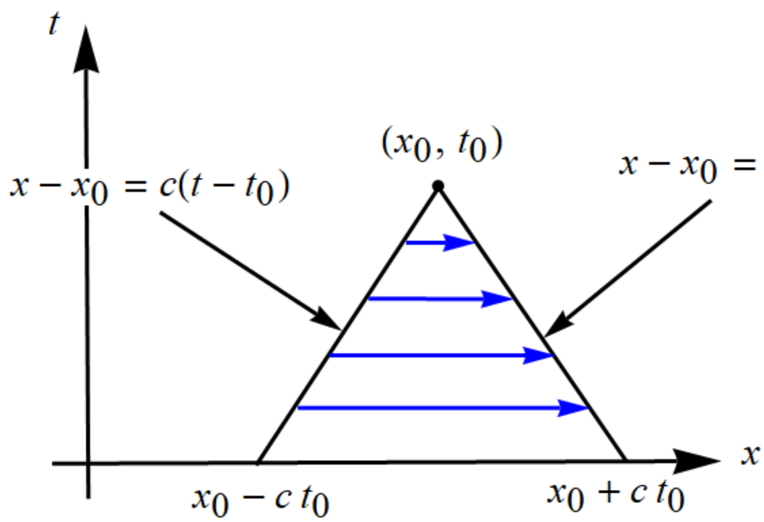
Solution by Green's Theorem

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x)$$

The characteristics were found to be straight lines, $\xi = x - ct$ and $\eta = x + ct$, with slopes $\pm c$. Suppose (x_0, t_0) is the point in the xt-plane we want to evaluate u at. The equations of the lines going through this point are

$$x - x_0 = c(t - t_0)$$
$$x - x_0 = -c(t - t_0).$$

Integrate both sides of the inhomogeneous wave equation over the triangular domain D enclosed by these lines (from left to right as indicated below).



Write the double integral explicitly on the right side.

$$\iint_D (u_{tt} - c^2 u_{xx})\, dA = \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\, dx\, dt$$

Rewrite the left side.

$$-\iint_D \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\, dx\, dt$$

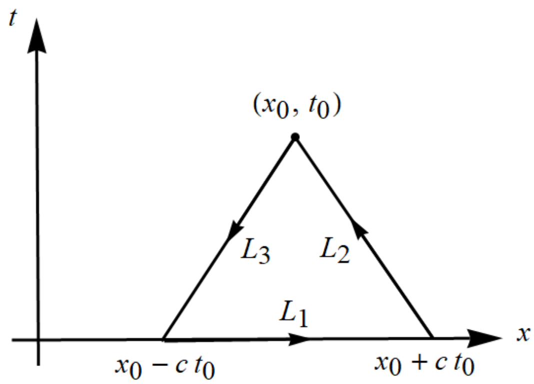
Multiply both sides by -1.

$$\iint_D \left[\frac{\partial}{\partial x}(c^2 u_x) - \frac{\partial}{\partial t}(u_t)\right] dA = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\, dx\, dt$$

Apply Green's theorem (essentially the divergence theorem in two dimensions) to the double integral on the left to turn it into a counterclockwise line integral around the triangle's boundary bdy D.

$$\oint_{\text{bdy } D} (u_t\, dx + c^2 u_x\, dt) = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\, dx\, dt$$

Let L_1, L_2, and L_3 represent the legs of the triangle.



The line integral is the sum of three integrals, one over each leg.

$$\int_{L_1} (u_t\,dx + c^2 u_x\,dt) + \int_{L_2} (u_t\,dx + c^2 u_x\,dt) + \int_{L_3} (u_t\,dx + c^2 u_x\,dt) = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\,dx\,dt$$

On L_1	On L_2	On L_3
$t = 0$	$x - x_0 = -c(t - t_0)$	$x - x_0 = c(t - t_0)$
$dt = 0$	$dx = -c\,dt$	$dx = c\,dt$

Replace the differentials in the integrals over L_2 and L_3.

$$\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\,dx + \int_{L_2} \left[u_t(-c\,dt) + c^2 u_x\left(-\frac{dx}{c}\right)\right] + \int_{L_3} \left[u_t(c\,dt) + c^2 u_x\left(\frac{dx}{c}\right)\right]$$
$$= -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\,dx\,dt$$

In this exercise $u_t(x,0) = \psi(x)$.

$$\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx - c\int_{L_2} \left(\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx\right) + c\int_{L_3} \left(\frac{\partial u}{\partial t}\,dt + \frac{\partial u}{\partial x}\,dx\right) = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\,dx\,dt$$

The second and third integrands are how the differential of $u = u(x,t)$ is defined.

$$\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx - c\int_{L_2} du + c\int_{L_3} du = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\,dx\,dt$$

Evaluate the second and third integrals.

$$\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx - c[u(x_0,t_0) - u(x_0+ct_0,0)] + c[u(x_0-ct_0,0) - u(x_0,t_0)] = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\,dx\,dt$$

In this exercise $u(x,0) = \phi(x)$, so $u(x_0 + ct_0, 0) = \phi(x_0 + ct_0)$ and $u(x_0 - ct_0, 0) = \phi(x_0 - ct_0)$.

$$\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx - 2cu(x_0,t_0) + c[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] = -\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x,t)\,dx\,dt$$

Solve this equation for $2cu(x_0, t_0)$.

$$2cu(x_0, t_0) = c[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] + \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\, dx + \int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x, t)\, dx\, dt$$

Divide both sides by $2c$.

$$u(x_0, t_0) = \frac{1}{2}[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\, dx + \frac{1}{2c}\int_0^{t_0} \int_{x_0+c(t-t_0)}^{x_0-c(t-t_0)} f(x, t)\, dx\, dt$$

Finally, switch the roles of x and t with those of x_0 and t_0, respectively.

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x_0)\, dx_0 + \frac{1}{2c}\int_0^{t} \int_{x+c(t_0-t)}^{x-c(t_0-t)} f(x_0, t_0)\, dx_0\, dt_0$$

Therefore, u is the sum of three terms, one each for f, ϕ, and ψ.

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x_0)\, dx_0 + \frac{1}{2c}\int_0^{t} \int_{x-c(t-t_0)}^{x+c(t-t_0)} f(x_0, t_0)\, dx_0\, dt_0$$

Solution by Duhamel's Principle

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad -\infty < x < \infty,\ t > 0$$
$$u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x)$$

Use the fact that the PDE is linear to split up the problem. Let $u(x,t) = v(x,t) + w(x,t)$, where v and w satisfy the following initial value problems.

$$\begin{array}{ll} v_{tt} - c^2 v_{xx} = 0 & w_{tt} - c^2 w_{xx} = f(x,t) \\ v(x,0) = \phi(x) \quad v_t(x,0) = \psi(x) & w(x,0) = 0 \quad w_t(x,0) = 0 \end{array}$$

The solution for v is given by d'Alembert's formula in section 2.1 on page 36.

$$v(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x_0)\,dx_0$$

According to Duhamel's principle, the solution to the inhomogeneous wave equation is

$$w(x,t) = \int_0^t W(x, t-s; s)\,ds,$$

where $W = W(x,t;s)$ is the solution to the associated homogeneous equation with a particular choice for the initial conditions.

$$W_{tt} - c^2 W_{xx} = 0, \quad -\infty < x < \infty,\ t > 0$$
$$W(x,0;s) = 0 \qquad W_t(x,0;s) = f(x,s)$$

The solution for W is given by d'Alembert's formula.

$$W(x,t;s) = \frac{1}{2c}\int_{x-ct}^{x+ct} f(r,s)\,dr$$

The solution to the inhomogeneous wave equation is then

$$\begin{aligned} w(x,t) &= \int_0^t \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(r,s)\,dr\,ds \\ &= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(r,s)\,dr\,ds. \end{aligned}$$

Therefore, u is the sum of three terms, one each for f, ϕ, and ψ.

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(x_0)\,dx_0 + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(r,s)\,dr\,ds.$$

We can check that the Duhamel solution satisfies the wave equation. Use the Leibnitz rule to differentiate the integrals.

$$\begin{aligned} w_{tt} - c^2 w_{xx} &= \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\int_0^t W(x,t-s;s)\,ds\right] - c^2 \frac{\partial^2}{\partial x^2}\int_0^t W(x,t-s;s)\,ds \\ &= \frac{\partial}{\partial t}\left[\int_0^t \frac{\partial}{\partial t} W(x,t-s;s)\,ds + \underbrace{W(x,0;t)}_{=0} \cdot 1 - W(x,t;0)\cdot 0\right] - c^2 \int_0^t W_{xx}(x,t-s;s)\,ds \\ &= \int_0^t \frac{\partial^2}{\partial t^2} W(x,t-s;s)\,ds + W_t(x,0;t)\cdot 1 - W_t(x,t;0)\cdot 0 - c^2\int_0^t W_{xx}(x,t-s;s)\,ds \\ &= \int_0^t \underbrace{[W_{tt}(x,t-s;s) - c^2 W_{xx}(x,t-s;s)]}_{=0}\,ds + W_t(x,0;t) = f(x,t) \end{aligned}$$