Exercise 5

Let $f(x,t)$ be any function and let $u(x,t) = (1/2c)\iint_\Delta f$, where Δ is the triangle of dependence. Verify directly by differentiation that

$$u_{tt} = c^2 u_{xx} + f \quad \text{and} \quad u(x,0) \equiv u_t(x,0) \equiv 0.$$

(*Hint:* Begin by writing the formula as the *iterated* integral

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)\,dy\,ds$$

and differentiate with care using the rule in the Appendix. This exercise is not easy.)

Solution

The aim here is to verify that

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)\,dy\,ds$$

satisfies the inhomogeneous wave equation, $u_{tt} = c^2 u_{xx} + f(x,t)$. In order to differentiate this double integral, it's necessary to use the Leibnitz rule, which states that if

$$I(t) = \int_{a(t)}^{b(t)} \gamma(x,t)\,dx,$$

then

$$\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial \gamma}{\partial t}\,dx + \gamma[b(t),t]b'(t) - \gamma[a(t),t]a'(t).$$

Apply the rule twice to obtain the first derivative of u with respect to t.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{1}{2c}\frac{\partial}{\partial t}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)\,dy\,ds \\
&= \frac{1}{2c}\left\{\int_0^t \left[\frac{\partial}{\partial t}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right] ds + \underbrace{\int_x^x f(y,t)\,dy}_{=0} \times 1 - \int_{x-ct}^{x+ct} f(y,0)\,dy \times 0\right\} \\
&= \frac{1}{2c}\int_0^t \left[\frac{\partial}{\partial t}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right] ds \\
&= \frac{1}{2c}\int_0^t \left\{\int_{x-c(t-s)}^{x+c(t-s)} \underbrace{\frac{\partial}{\partial t} f(y,s)}_{=0}\,dy + f[x+c(t-s),s] \times (c) - f[x-c(t-s),s] \times (-c)\right\} ds
\end{aligned}$$

Thus,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\int_0^t \{f[x+c(t-s),s] + f[x-c(t-s),s]\}\,ds.$$

Apply the rule once more to obtain the second derivative of u with respect to t.

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \frac{1}{2}\frac{\partial}{\partial t}\int_0^t \{f[x+c(t-s),s]+f[x-c(t-s),s]\}\,ds \\ &= \frac{1}{2}\left\{\int_0^t \frac{\partial}{\partial t}\{f[x+c(t-s),s]+f[x-c(t-s),s]\}\,ds \right. \\ &\qquad \left. + [f(x,t)+f(x,t)]\times 1 - [f(x+ct,0)+f(x-ct,0)]\times 0\right\} \\ &= \frac{1}{2}\int_0^t \{f_a[x+c(t-s),s]\times(c)+f_a[x-c(t-s),s]\times(-c)\}\,ds + f(x,t),\end{aligned}$$

where f_a is a derivative of f with respect to its first argument. Thus,

$$\frac{\partial^2 u}{\partial t^2} = \frac{c}{2}\int_0^t \{f_a[x+c(t-s),s]-f_a[x-c(t-s),s]\}\,ds + f(x,t).$$

Now we will take derivatives of u with respect to x. We have the following for the first derivative.

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{1}{2c}\frac{\partial}{\partial x}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)\,dy\,ds \\ &= \frac{1}{2c}\int_0^t \left[\frac{\partial}{\partial x}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\right] ds \\ &= \frac{1}{2c}\int_0^t \left\{\int_{x-c(t-s)}^{x+c(t-s)} \underbrace{\frac{\partial}{\partial x} f(y,s)}_{=0}\,dy + f[x+c(t-s),s]\times 1 - f[x-c(t-s),s]\times 1\right\} ds\end{aligned}$$

Thus,

$$\frac{\partial u}{\partial x} = \frac{1}{2c}\int_0^t \{f[x+c(t-s),s]-f[x-c(t-s),s]\}\,ds.$$

Now the second derivative of u with respect to x will be obtained.

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{1}{2c}\frac{\partial}{\partial x}\int_0^t \{f[x+c(t-s),s]-f[x-c(t-s),s]\}\,ds \\ &= \frac{1}{2c}\int_0^t \frac{\partial}{\partial x}\{f[x+c(t-s),s]-f[x-c(t-s),s]\}\,ds \\ &= \frac{1}{2c}\int_0^t \{f_a[x+c(t-s),s]\times 1 - f_a[x-c(t-s),s]\times 1\}\,ds,\end{aligned}$$

where f_a is a derivative of f with respect to its first argument. Thus,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2c}\int_0^t \{f_a[x+c(t-s),s]-f_a[x-c(t-s),s]\}\,ds.$$

Therefore,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + f(x,t)$$

and

$$u(x,0) = \frac{1}{2c}\int_0^0 \int_{x+cs}^{x-cs} f(y,s)\,dy\,ds = 0 \quad \text{and} \quad u_t(x,0) = \frac{1}{2}\int_0^0 \{f[x+c(t-s),s]+f[x-c(t-s),s]\}\,ds = 0.$$