Exercise 8

Show that the source operator for the wave equation solves the problem

$$\mathscr{S}_{tt} - c^2\mathscr{S}_{xx} = 0, \quad \mathscr{S}(0) = 0, \quad \mathscr{S}_t(0) = I,$$

where I is the identity operator.

Solution

Apply both sides of each equation to $\psi(x)$.

$$(\mathscr{S}_{tt} - c^2\mathscr{S}_{xx})\psi(x) = 0, \quad \mathscr{S}(0)\psi(x) = 0, \quad \mathscr{S}_t(0)\psi(x) = \psi(x)$$

Start by showing that the source operator satisfies the PDE, using the Leibnitz rule to differentiate the integrals.

$$\begin{aligned}
(\mathscr{S}_{tt} - c^2\mathscr{S}_{xx})\psi(x) &= \left[\frac{\partial^2}{\partial t^2}\left(\frac{1}{2c}\int_{x-ct}^{x+ct}(\cdot)\,dr\right) - c^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{2c}\int_{x-ct}^{x+ct}(\cdot)\,dr\right)\right]\psi(x) \\
&= \frac{\partial^2}{\partial t^2}\left[\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(r)\,dr\right] - c^2\frac{\partial^2}{\partial x^2}\left[\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(r)\,dr\right] \\
&= \frac{1}{2c}\frac{\partial^2}{\partial t^2}\left[\int_{x-ct}^{x+ct}\psi(r)\,dr\right] - \frac{c}{2}\frac{\partial^2}{\partial x^2}\left[\int_{x-ct}^{x+ct}\psi(r)\,dr\right] \\
&= \frac{1}{2c}\frac{\partial}{\partial t}\left[\int_{x-ct}^{x+ct}\underbrace{\frac{\partial}{\partial t}\psi(r)}_{=0}\,dr + \psi(x+ct)\cdot c - \psi(x-ct)\cdot(-c)\right] \\
&\qquad - \frac{c}{2}\frac{\partial}{\partial x}\left[\int_{x-ct}^{x+ct}\underbrace{\frac{\partial}{\partial x}\psi(r)}_{=0}\,dr + \psi(x+ct)\cdot 1 - \psi(x-ct)\cdot 1\right] \\
&= \frac{1}{2}\frac{\partial}{\partial t}[\psi(x+ct) + \psi(x-ct)] - \frac{c}{2}\frac{\partial}{\partial x}[\psi(x+ct) - \psi(x-ct)] \\
&= \frac{1}{2}[\psi'(x+ct)\cdot c + \psi'(x-ct)\cdot(-c)] - \frac{c}{2}[\psi'(x+ct)\cdot 1 - \psi'(x-ct)\cdot 1] \\
&= \frac{c}{2}[\cancel{\psi'(x+ct)} - \cancel{\psi'(x-ct)}] - \frac{c}{2}[\cancel{\psi'(x+ct)} - \cancel{\psi'(x-ct)}] \\
&= 0
\end{aligned}$$

Now show that the source operator satisfies the first initial condition.

$$\begin{aligned}
\mathscr{S}(0)\psi(x) &= \left[\frac{1}{2c}\int_{x-c(0)}^{x+c(0)}(\cdot)\,dr\right]\psi(x) \\
&= \frac{1}{2c}\int_{x}^{x}\psi(r)\,dr \\
&= 0
\end{aligned}$$

Now show that the source operator satisfies the second initial condition.

$$
\begin{aligned}
\mathscr{S}_t(0)\psi(x) &= \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{x-ct}^{x+ct}(\cdot)\,dr\right]\bigg|_{t=0}\psi(x) \\
&= \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{x-ct}^{x+ct}\psi(r)\,dr\right]\bigg|_{t=0} \\
&= \frac{1}{2c}\left[\frac{\partial}{\partial t}\int_{x-ct}^{x+ct}\psi(r)\,dr\right]\bigg|_{t=0} \\
&= \frac{1}{2c}\left[\int_{x-ct}^{x+ct}\underbrace{\frac{\partial}{\partial t}\psi(r)}_{=\,0}\,dr + \psi(x+ct)\cdot c - \psi(x-ct)\cdot(-c)\right]\bigg|_{t=0} \\
&= \frac{1}{2}[\psi(x+ct)+\psi(x-ct)]\bigg|_{t=0} \\
&= \frac{1}{2}[2\psi(x)] \\
&= \psi(x)
\end{aligned}
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