Exercise 1

(a) Use the Fourier expansion to explain why the note produced by a violin string rises sharply by one octave when the string is clamped exactly at its midpoint.

(b) Explain why the note rises when the string is tightened.

Solution

Part (a)

The Fourier series solution to the wave equation,

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \quad \text{for } 0 < x < l \\ u(0,t) &= 0 = u(l,t), \end{aligned}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi ct}{l} + B_n \cos \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}.$$

The frequency of the violin note is given by the coefficient of t. (The coefficient of x gives information about the wavelength.)

$$\nu_n = \frac{n\pi c}{l}, \quad n = 1, 2, \ldots$$

For a string,

$$c^2 = \frac{T}{\rho},$$

where T is the tension of the string and ρ is its mass density.

$$\nu_n = \frac{n\pi}{l}\sqrt{\frac{T}{\rho}}, \quad n = 1, 2, \ldots$$

In the event that the string gets clamped at the midpoint, the length l gets cut in half. The frequency doubles as a result, i.e. the note rises by one octave.

$$l \quad \rightarrow \quad \frac{l}{2} \quad \Rightarrow \quad \frac{n\pi}{l}\sqrt{\frac{T}{\rho}} \quad \rightarrow \quad 2\frac{n\pi}{l}\sqrt{\frac{T}{\rho}}$$

Part (b)

When the string is tightened, the tension increases. Since the frequency is proportional to the square root of tension, the note rises when this happens.