Exercise 4

Consider waves in a resistant medium that satisfy the problem

$$u_{tt} = c^2 u_{xx} - r u_t \quad \text{for } 0 < x < l$$
$$u = 0 \quad \text{at both ends}$$
$$u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x),$$

where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.

Solution

The PDE and its boundary conditions,

$$u(0,t) = 0$$
$$u(l,t) = 0,$$

are linear and homogeneous, so the method of separation of variables can be applied to solve it. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and plug it into the PDE

$$u_{tt} = c^2 u_{xx} - r u_t \quad \rightarrow \quad XT'' = c^2 X'' T - r X T'$$

and the boundary conditions.

$$u(0,t) = X(0)T(t) = 0 \quad \rightarrow \quad X(0) = 0$$
$$u(l,t) = X(l)T(t) = 0 \quad \rightarrow \quad X(l) = 0$$

Separate variables now.

$$XT'' + rXT' = c^2 X'' T \quad \rightarrow \quad \frac{T'' + rT'}{c^2 T} = \frac{X''}{X}$$

Note that c^2 is a constant and can go on either side. The final answer will be the same regardless. We have a function of t on the left side and a function of x on the right side. The only way both functions can be equal is if they are equal to a constant.

$$\frac{T'' + rT'}{c^2 T} = \frac{X''}{X} = k$$

Values of k for which $X(0) = 0$ and $X(l) = 0$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $k = \mu^2$

Assuming k is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Now use the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(0) &= C_1 = 0 \\ X(l) &= C_1 \cosh \mu l + C_2 \sinh \mu l = 0 \end{aligned}$$

We see that $C_1 = 0$ and $C_2 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering positive values for k, and there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $k = 0$

Assuming k is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_3 x + C_4$$

Now use the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(0) &= C_4 = 0 \\ X(l) &= C_3 l + C_4 = 0 \end{aligned}$$

We see that $C_3 = 0$ and $C_4 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering $k = 0$, and zero is not an eigenvalue.

Determination of Negative Eigenvalues: $k = -\lambda^2$

Assuming k is negative, the differential equation for X becomes

$$\frac{X''}{X} = -\lambda^2.$$

Multiply both sides by X.

$$X'' = -\lambda^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \lambda x + C_6 \sin \lambda x$$

Now use the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(0) &= C_5 = 0 \\ X(l) &= C_5 \cos \lambda l + C_6 \sin \lambda l = 0 \end{aligned}$$

The second equation simplifies to $C_6 \sin \lambda l = 0$. To avoid getting the trivial solution, we insist that $C_6 \neq 0$. Doing so yields an equation for the eigenvalues.

$$\sin \lambda l = 0$$

Solve for λl.

$$\lambda l = n\pi, \quad n = 1, 2, \ldots$$

So then

$$\lambda = \lambda_n = \frac{n\pi}{l}, \quad n = 1, 2, \ldots.$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_6 \sin \lambda x \quad \rightarrow \quad X_n(x) = \sin \lambda_n x, \quad n = 1, 2, \ldots.$$

Now solve the differential equation for $T(t)$.

$$\frac{T'' + rT'}{c^2 T} = -\lambda^2$$

Multiply both sides by $c^2 T$.

$$T'' + rT' = -c^2 \lambda^2 T$$

Bring $c^2 \lambda^2 T$ to the left side.

$$T'' + rT' + c^2 \lambda^2 T = 0$$

This is an ODE with constant coefficients, so its solution is of the form

$$T = e^{st}.$$

Substitute this into the ODE to determine s.

$$s^2 e^{st} + rse^{st} + c^2 \lambda^2 e^{st} = 0$$

Divide both sides by e^{st}.

$$s^2 + rs + c^2 \lambda^2 = 0$$

This is a quadratic equation for s, so use the quadratic formula to solve for it.

$$s = \frac{-r \pm \sqrt{r^2 - 4c^2 \lambda^2}}{2}$$

Now substitute $\lambda = n\pi/l$.

$$\begin{aligned} s &= \frac{-r \pm \sqrt{r^2 - 4c^2 \left(\frac{n\pi}{l}\right)^2}}{2} \\ &= \frac{-r \pm \sqrt{r^2 - n^2 \left(\frac{2\pi c}{l}\right)^2}}{2} \end{aligned}$$

Since $0 < r < 2\pi c/l$, the quantity under the square root is negative for every value that n takes. Factor out -1 and bring it out of the square root as i.

$$\begin{aligned} s &= \frac{-r \pm i\sqrt{n^2 \left(\frac{2\pi c}{l}\right)^2 - r^2}}{2} \\ &= -\frac{r}{2} \pm i\sqrt{\frac{n^2 \pi^2 c^2}{l^2} - \frac{r^2}{4}} \\ &= -\frac{r}{2} \pm i\frac{\sqrt{4n^2 \pi^2 c^2 - r^2 l^2}}{2l} \end{aligned}$$

Thus, the general solution to the ODE for T is

$$T(t) = C_7 e^{-\frac{r}{2}t} \cos\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right) + C_8 e^{-\frac{r}{2}t} \sin\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right).$$

According to the principle of linear superposition, the solution to the PDE for $u(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n e^{-\frac{r}{2}t} \cos\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right) + B_n e^{-\frac{r}{2}t} \sin\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right)\right] \sin\frac{n\pi x}{l}$$

Therefore,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{r}{2}t}\left[A_n \cos\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right) + B_n \sin\left(\frac{\sqrt{4n^2\pi^2c^2 - r^2l^2}}{2l}t\right)\right] \sin\frac{n\pi x}{l}.}$$

The final task is to use Fourier's method to express the coefficients, A_n and B_n, in terms of the provided initial data, $\phi(x)$ and $\psi(x)$.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} = \phi(x)$$

Multiply both sides by $\sin\lambda_m x$, where m is a positive integer.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} = \phi(x)\sin\frac{m\pi x}{l}$$

Integrate both sides with respect to x over the domain the PDE is defined.

$$\int_0^l \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx$$

Bring the integral inside the sum on the left.

$$\sum_{n=1}^{\infty} A_n \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \int_0^l \phi(x)\sin\frac{m\pi x}{l}\,dx$$

Since n and m are integers,

$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \begin{cases} \frac{l}{2} & n = m \\ 0 & n \neq m \end{cases},$$

as can be verified with trigonometric identities. This implies that every term in the infinite series vanishes except for the $n = m$ term.

$$A_n \cdot \frac{l}{2} = \int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx$$

Therefore, the first coefficient is

$$\boxed{A_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}\,dx.}$$

In order to use the second initial condition, differentiate the boxed solution for u with respect to t.

$$u_t(x,t) = \sum_{n=1}^{\infty}\left(-\frac{r}{2}\right)e^{-\frac{r}{2}t}\left[A_n\cos\left(\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}t\right)+B_n\sin\left(\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}t\right)\right]\sin\frac{n\pi x}{l}$$
$$+\sum_{n=1}^{\infty}e^{-\frac{r}{2}t}\left[-A_n\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}\sin\left(\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}t\right)\right.$$
$$\left.+B_n\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}\cos\left(\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}t\right)\right]\sin\frac{n\pi x}{l}$$

Now plug in $t=0$.

$$u_t(x,0) = \sum_{n=1}^{\infty}\left(-\frac{r}{2}\right)A_n\sin\frac{n\pi x}{l}+\sum_{n=1}^{\infty}B_n\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}\sin\frac{n\pi x}{l}=\psi(x)$$

Bring the constant in front of the first sum.

$$\left(-\frac{r}{2}\right)\sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{l}+\sum_{n=1}^{\infty}B_n\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}\sin\frac{n\pi x}{l}=\psi(x)$$

The first sum is $u(x,0)$, and $u(x,0)$ is equal to $\phi(x)$.

$$\left(-\frac{r}{2}\right)\phi(x)+\sum_{n=1}^{\infty}B_n\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}\sin\frac{n\pi x}{l}=\psi(x)$$

Bring the first term to the right side.

$$\sum_{n=1}^{\infty}B_n\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}\sin\frac{n\pi x}{l}=\frac{r}{2}\phi(x)+\psi(x)$$

Now multiply both sides by $\sin\lambda_m x$, where m is a positive integer.

$$\sum_{n=1}^{\infty}B_n\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}=\left[\frac{r}{2}\phi(x)+\psi(x)\right]\sin\frac{m\pi x}{l}$$

Integrate both sides with respect to x over the domain the PDE is defined.

$$\int_0^l\sum_{n=1}^{\infty}B_n\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx=\int_0^l\left[\frac{r}{2}\phi(x)+\psi(x)\right]\sin\frac{m\pi x}{l}\,dx$$

Bring the integral inside the sum on the left.

$$\sum_{n=1}^{\infty}B_n\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}\int_0^l\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx=\int_0^l\left[\frac{r}{2}\phi(x)+\psi(x)\right]\sin\frac{m\pi x}{l}\,dx$$

Every term in this infinite series vanishes except for the one when $n=m$.

$$B_n\frac{\sqrt{4n^2\pi^2c^2-r^2l^2}}{2l}\cdot\frac{l}{2}=\int_0^l\left[\frac{r}{2}\phi(x)+\psi(x)\right]\sin\frac{n\pi x}{l}\,dx$$

Therefore,

$$\boxed{B_n=\frac{4}{\sqrt{4n^2\pi^2c^2-r^2l^2}}\int_0^l\left[\frac{r}{2}\phi(x)+\psi(x)\right]\sin\frac{n\pi x}{l}\,dx.}$$