

## Exercise 1

Solve the diffusion problem  $u_t = ku_{xx}$  in  $0 < x < l$ , with the mixed boundary conditions  $u(0, t) = u_x(l, t) = 0$ .

### Solution

Since the diffusion equation and its boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form,  $u(x, t) = X(x)T(t)$ , and plug it into the PDE

$$u_t = ku_{xx} \quad \rightarrow \quad XT' = kX''T$$

and the boundary conditions.

$$\begin{aligned} u(0, t) = X(0)T(t) = 0 & \quad \rightarrow \quad X(0) = 0 \\ u_x(l, t) = X'(l)T(t) = 0 & \quad \rightarrow \quad X'(l) = 0 \end{aligned}$$

Separate variables now.

$$\frac{T'}{kT} = \frac{X''}{X}$$

Note that  $k$  is a constant and can go on either side. The final answer will be the same regardless. We have a function of  $t$  on the left side and a function of  $x$  on the right side. The only way both functions can be equal is if they are equal to a constant.

$$\frac{T'}{kT} = \frac{X''}{X} = p$$

Values of  $p$  for which  $X(0) = 0$  and  $X'(l) = 0$  are satisfied are called the eigenvalues, and the nontrivial functions  $X(x)$  associated with them are called the eigenfunctions.

### Determination of Positive Eigenvalues: $p = \mu^2$

Assuming  $p$  is positive, the differential equation for  $X$  becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by  $X$ .

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Now use the boundary conditions to determine  $C_1$  and  $C_2$ .

$$\begin{aligned} X(0) = C_1 &= 0 \\ X'(l) = C_1 \mu \sinh \mu l + C_2 \mu \cosh \mu l &= 0 \end{aligned}$$

We see that  $C_1 = 0$  and  $C_2 = 0$ . Hence, only the trivial solution  $X(x) = 0$  results from considering positive values for  $p$ , and there are no positive eigenvalues.

**Determination of the Zero Eigenvalue:  $p = 0$** 

Assuming  $p$  is zero, the differential equation for  $X$  becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by  $X$ .

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_3x + C_4$$

Now use the boundary conditions to determine  $C_3$  and  $C_4$ .

$$X(0) = C_4 = 0$$

$$X'(l) = C_3 = 0$$

We see that  $C_3 = 0$  and  $C_4 = 0$ . Hence, only the trivial solution  $X(x) = 0$  results from considering  $p = 0$ , and zero is not an eigenvalue.

**Determination of Negative Eigenvalues:  $p = -\lambda^2$** 

Assuming  $p$  is negative, the differential equation for  $X$  becomes

$$\frac{X''}{X} = -\lambda^2.$$

Multiply both sides by  $X$ .

$$X'' = -\lambda^2 X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \lambda x + C_6 \sin \lambda x$$

Now use the boundary conditions to determine  $C_5$  and  $C_6$ .

$$X(0) = C_5 = 0$$

$$X'(l) = -C_5 \lambda \sin \lambda l + C_6 \lambda \cos \lambda l = 0$$

The second equation simplifies to  $C_6 \lambda \cos \lambda l = 0$ . To avoid getting the trivial solution, we insist that  $C_6 \neq 0$ . Doing so yields an equation for the eigenvalues.

$$\cos \lambda l = 0$$

Solve for  $\lambda l$ .

$$\lambda l = \frac{1}{2}(2n + 1)\pi, \quad n = 0, 1, \dots$$

So then

$$\lambda = \lambda_n = \frac{\pi}{2l}(2n + 1), \quad n = 0, 1, \dots$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_6 \sin \lambda x \quad \rightarrow \quad X_n(x) = \sin \lambda_n x, \quad n = 0, 1, \dots$$

Now solve the differential equation for  $T(t)$ .

$$\frac{T'}{kT} = -\lambda^2$$

Multiply both sides by  $k$ .

$$\frac{T'}{T} = -k\lambda^2$$

The left side is just the derivative of  $\ln T$ .

$$\frac{d}{dt}(\ln T) = -k\lambda^2$$

Integrate both sides with respect to  $t$ .

$$\ln T = -k\lambda^2 t + C_7$$

Exponentiate both sides.

$$T(t) = e^{-k\lambda^2 t + C_7} = e^{-k\lambda^2 t} e^{C_7}$$

Use a new constant of integration.

$$T(t) = C_8 e^{-k\lambda^2 t} \quad \rightarrow \quad T_n(t) = e^{-k\lambda_n^2 t}$$

According to the principle of linear superposition, the solution to the PDE for  $u(x, t)$  is a linear combination of all products  $T_n(t)X_n(x)$  over all the eigenvalues.

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} A_n T_n(t) X_n(x) \\ &= \sum_{n=0}^{\infty} A_n e^{-k\lambda_n^2 t} \sin \lambda_n x \\ &= \sum_{n=0}^{\infty} A_n e^{-k\left[\frac{\pi}{2l}(2n+1)\right]^2 t} \sin \left[\frac{\pi}{2l}(2n+1)x\right] \end{aligned}$$

Therefore,

$$u(x, t) = \sum_{n=0}^{\infty} A_n \exp \left[ -k \frac{\pi^2}{4l^2} (2n+1)^2 t \right] \sin \left[ \frac{\pi}{2l} (2n+1)x \right].$$

If an initial condition were provided,  $A_n$  could be determined.