Exercise 4

Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \le x \le l$. Then the concentration of the diffusing substance satisfies

$$u_t = ku_{xx} \quad \text{for } -l \le x \le l$$
$$u(-l,t) = u(l,t) \quad \text{and} \quad u_x(-l,t) = u_x(l,t).$$

These are called *periodic boundary conditions*.

(a) Show that the eigenvalues are $\lambda = (n\pi/l)^2$ for $n = 0, 1, 2, 3, \ldots$.

(b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right)e^{-n^2\pi^2 kt/l^2}.$$

Solution

Since the diffusion equation and its boundary conditions are linear and homogeneous, the method of separation of variables can be applied to solve it. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and plug it into the PDE

$$u_t = ku_{xx} \quad \to \quad XT' = kX''T$$

and the boundary conditions.

$$\begin{aligned} u(-l,t) = u(l,t) \quad &\to \quad X(-l)T(t) = X(l)T(t) \quad \to \quad X(-l) = X(l) \\ u_x(-l,t) = u_x(l,t) \quad &\to \quad X'(-l)T(t) = X'(l)T(t) \quad \to \quad X'(-l) = X'(l) \end{aligned}$$

Separate variables now.

$$\frac{T'}{kT} = \frac{X''}{X}$$

Note that k is a constant and can go on either side. The final answer will be the same regardless. We have a function of t on the left side and a function of x on the right side. The only way both functions can be equal is if they are equal to a constant.

$$\frac{T'}{kT} = \frac{X''}{X} = p$$

Values of p for which $X(-l) = X(l)$ and $X'(-l) = X'(l)$ are satisfied are called the eigenvalues, and the nontrivial functions $X(x)$ associated with them are called the eigenfunctions.

Determination of Positive Eigenvalues: $p = \mu^2$

Assuming p is positive, the differential equation for X becomes

$$\frac{X''}{X} = \mu^2.$$

Multiply both sides by X.

$$X'' = \mu^2 X$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

Now use the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X(-l) = X(l) \quad &\rightarrow \quad \cancel{C_1 \cosh \mu l} - C_2 \sinh \mu l = \cancel{C_1 \cosh \mu l} + C_2 \sinh \mu l \\ X'(-l) = X'(l) \quad &\rightarrow \quad -C_1 \mu \sinh \mu l + \cancel{C_2 \mu \cosh \mu l} = C_1 \mu \sinh \mu l + \cancel{C_2 \mu \cosh \mu l} \end{aligned}$$

Solving this system of equations yields $C_1 = 0$ and $C_2 = 0$. Hence, only the trivial solution $X(x) = 0$ results from considering positive values for p, and there are no positive eigenvalues.

Determination of the Zero Eigenvalue: $p = 0$

Assuming p is zero, the differential equation for X becomes

$$\frac{X''}{X} = 0.$$

Multiply both sides by X.

$$X'' = 0$$

The general solution is a linear function.

$$X(x) = C_3 x + C_4$$

Now use the boundary conditions to determine C_3 and C_4.

$$\begin{aligned} X(-l) = X(l) \quad &\rightarrow \quad -C_3 l + \cancel{C_4} = C_3 l + \cancel{C_4} \\ X'(-l) = X'(l) \quad &\rightarrow \quad C_3 = C_3 \end{aligned}$$

We see that $C_3 = 0$ and C_4 is arbitrary. Hence, zero is an eigenvalue, and $X(x) = C_4$ is the eigenfunction associated with it. Now solve the differential equation for $T(t)$.

$$\frac{T'}{kT} = 0$$

Multiply both sides by kT.

$$T' = 0$$

From this we conclude that $T(t)$ is a constant as well.

Determination of Negative Eigenvalues: $p = -\lambda$

In order to be consistent with Mr. Strauss's notation, use $-\lambda$ rather than $-\lambda^2$ for p, where $\lambda > 0$. Doing this makes no difference in the final answer for $u(x,t)$. The differential equation for X becomes

$$\frac{X''}{X} = -\lambda.$$

Multiply both sides by X.

$$X'' = -\lambda X$$

The general solution can be written in terms of sine and cosine.

$$X(x) = C_5 \cos \sqrt{\lambda} x + C_6 \sin \sqrt{\lambda} x$$

Now use the boundary conditions to determine C_5 and C_6.

$$\begin{aligned} X(-l) = X(l) \quad &\rightarrow \quad \cancel{C_5 \cos \sqrt{\lambda} l} - C_6 \sin \sqrt{\lambda} l = \cancel{C_5 \cos \sqrt{\lambda} l} + C_6 \sin \sqrt{\lambda} l \\ X'(-l) = X'(l) \quad &\rightarrow \quad C_5 \sqrt{\lambda} \sin \sqrt{\lambda} l + \cancel{C_6 \sqrt{\lambda} \cos \sqrt{\lambda} l} = -C_5 \sqrt{\lambda} \sin \sqrt{\lambda} l + \cancel{C_6 \sqrt{\lambda} \cos \sqrt{\lambda} l} \end{aligned}$$

The system of equations for C_5 and C_6 is

$$\begin{aligned} 2C_6 \sin \sqrt{\lambda} l &= 0 \\ 2C_5 \sqrt{\lambda} \sin \sqrt{\lambda} l &= 0. \end{aligned}$$

These two equations imply that C_5 and C_6 can be arbitrary as long as $\sin \sqrt{\lambda} l = 0$. From this equation λ can be solved for.

$$\sqrt{\lambda} l = n\pi, \quad n = 1, 2, \ldots$$

So then

$$\lambda = \lambda_n = \frac{n^2 \pi^2}{l^2}, \quad n = 1, 2, \ldots.$$

The eigenfunctions associated with these eigenvalues are

$$X(x) = C_5 \cos \sqrt{\lambda} x + C_6 \sin \sqrt{\lambda} x.$$

Therefore, all the eigenvalues for this boundary value problem are

$$\boxed{0 \quad \text{and} \quad \lambda = \frac{n^2 \pi^2}{l^2}, \quad n = 1, 2, \ldots.}$$

Now solve the differential equation for $T(t)$.

$$\frac{T'}{kT} = -\lambda$$

Multiply both sides by k.

$$\frac{T'}{T} = -k\lambda$$

The left side is just the derivative of $\ln T$.

$$\frac{d}{dt}(\ln T) = -k\lambda$$

Integrate both sides with respect to t.

$$\ln T = -k\lambda t + C_7$$

Exponentiate both sides.

$$T(t) = e^{-k\lambda t + C_7} = e^{-k\lambda t} e^{C_7}$$

Use a new constant of integration.

$$T(t) = C_8 e^{-k\lambda t} \quad \rightarrow \quad T_n(t) = e^{-k\lambda_n t}$$

According to the principle of linear superposition, the solution to the PDE for $u(x,t)$ is a linear combination of all products $T_n(t)X_n(x)$ over all the eigenvalues.

$$\begin{aligned} u(x,t) &= C_0 + \sum_{n=1}^{\infty} e^{-k\lambda_n t}(A_n \cos\sqrt{\lambda_n}x + B_n \sin\sqrt{\lambda_n}x) \\ &= C_0 + \sum_{n=1}^{\infty} e^{-k\frac{n^2\pi^2}{l^2}t}\left(A_n \cos\sqrt{\frac{n^2\pi^2}{l^2}}x + B_n \sin\sqrt{\frac{n^2\pi^2}{l^2}}x\right) \end{aligned}$$

For the zero eigenvalue, $T(t)$ and $X(x)$ were found to be constants; consequently, anything (except functions of x or t) can be put in the place of C_0 since it is arbitrary. Therefore,

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \exp\left(-k\frac{n^2\pi^2}{l^2}t\right)\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right).$$

If an initial condition were provided, A_0, A_n, and B_n could be determined.