Exercise 11

(a) Prove that the (total) energy is conserved for the wave equation with Dirichlet BCs, where the energy is defined to be

$$E = \frac{1}{2}\int_0^l (c^{-2}u_t^2 + u_x^2)\,dx.$$

(Compare this definition with Section 2.2.)

(b) Do the same for the Neumann BCs.

(c) For the Robin BCs, show that

$$E_R = \frac{1}{2}\int_0^l (c^{-2}u_t^2 + u_x^2)\,dx + \frac{1}{2}a_l[u(l,t)]^2 + \frac{1}{2}a_0[u(0,t)]^2$$

is conserved. Thus, while the total energy E_R is still a constant, some of the internal energy is "lost" to the boundary if a_0 and a_l are positive and "gained" from the boundary if a_0 and a_l are negative.

Solution

The wave equation is

$$u_{tt} = c^2 u_{xx}.$$

Multiply both sides of the wave equation by u_t.

$$u_t u_{tt} = c^2 u_t u_{xx}$$

Divide both sides by c^2.

$$c^{-2} u_t u_{tt} = u_t u_{xx}$$

Rewrite the left side of the equation as follows.

$$\frac{1}{2}\frac{\partial}{\partial t}(c^{-2}u_t^2) = u_t u_{xx}$$

Now integrate both sides with respect to x over the domain that the wave equation is defined.

$$\int_0^l \frac{1}{2}\frac{\partial}{\partial t}(c^{-2}u_t^2)\,dx = \int_0^l u_t u_{xx}\,dx$$

Bring 1/2 and the partial derivative in front of the integral. Because the integral is definite over x, the result is function of t only. Thus, an ordinary derivative is used.

$$\frac{1}{2}\frac{d}{dt}\int_0^l c^{-2}u_t^2\,dx = \int_0^l u_t u_{xx}\,dx$$

Use integration by parts for the integral on the right side.

$$\frac{1}{2}\frac{d}{dt}\int_0^l c^{-2}u_t^2\,dx = u_t u_x\bigg|_0^l - \int_0^l u_{tx}u_x\,dx$$

Bring the integral on the right side to the left.

$$\frac{1}{2}\frac{d}{dt}\int_0^l c^{-2}u_t^2\,dx + \int_0^l u_{tx}u_x\,dx = u_t u_x\Big|_0^l$$

Rewrite the integrand of the second integral as follows.

$$\frac{1}{2}\frac{d}{dt}\int_0^l c^{-2}u_t^2\,dx + \int_0^l \frac{1}{2}\frac{\partial}{\partial t}(u_x^2)\,dx = u_t u_x\Big|_0^l$$

Bring 1/2 and the partial derivative in front of the integral. Because the integral is definite over x, the result is function of t only. Thus, an ordinary derivative is used.

$$\frac{1}{2}\frac{d}{dt}\int_0^l c^{-2}u_t^2\,dx + \frac{1}{2}\frac{d}{dt}\int_0^l u_x^2\,dx = u_t u_x\Big|_0^l$$

Combine the two terms on the left side.

$$\frac{1}{2}\frac{d}{dt}\int_0^l (c^{-2}u_t^2 + u_x^2)\,dx = u_t u_x\Big|_0^l$$

Here we use the definition of E.

$$\frac{dE}{dt} = u_t u_x\Big|_0^l$$

Finally, expand the right side.

$$\frac{dE}{dt} = u_t(l,t)u_x(l,t) - u_t(0,t)u_x(0,t)$$

Part (a)

If the wave equation is subjected to Dirichlet boundary conditions, then we have

$$u(0,t) = 0 \quad \text{and} \quad u(l,t) = 0.$$

Differentiate both sides of each equation with respect to t.

$$u_t(0,t) = 0 \quad \text{and} \quad u_t(l,t) = 0.$$

Therefore,

$$\frac{dE}{dt} = \underbrace{u_t(l,t)}_{=\,0}u_x(l,t) - \underbrace{u_t(0,t)}_{=\,0}u_x(0,t) = 0,$$

which means that energy is conserved when there are Dirichlet boundary conditions.

Part (b)

If the wave equation is subjected to Neumann boundary conditions, then we have

$$u_x(0,t) = 0 \quad \text{and} \quad u_x(l,t) = 0.$$

Therefore,

$$\frac{dE}{dt} = u_t(l,t)\underbrace{u_x(l,t)}_{=\,0} - u_t(0,t)\underbrace{u_x(0,t)}_{=\,0} = 0,$$

which means that energy is conserved when there are Neumann boundary conditions.

Part (c)

If the wave equation is subjected to Robin boundary conditions, then we have

$$u_x(0,t) - a_0 u(0,t) = 0 \quad \text{and} \quad u_x(l,t) + a_l u(l,t) = 0.$$

Solve each equation for $u_x(0,t)$ and $u_x(l,t)$.

$$u_x(0,t) = a_0 u(0,t) \quad \text{and} \quad u_x(l,t) = -a_l u(l,t)$$

So we have

$$\begin{aligned}
\frac{dE}{dt} &= u_t(l,t)u_x(l,t) - u_t(0,t)u_x(0,t) \\
&= u_t(l,t)[-a_l u(l,t)] - u_t(0,t)[a_0 u(0,t)] \\
&= -a_l u(l,t)u_t(l,t) - a_0 u(0,t)u_t(0,t) \\
&= -a_l \left\{\frac{1}{2}\frac{d}{dt}[u(l,t)]^2\right\} - a_0 \left\{\frac{1}{2}\frac{d}{dt}[u(0,t)]^2\right\} \\
&= -\frac{1}{2}a_l \frac{d}{dt}[u(l,t)]^2 - \frac{1}{2}a_0 \frac{d}{dt}[u(0,t)]^2.
\end{aligned}$$

Bring all terms to the left side.

$$\frac{dE}{dt} + \frac{1}{2}a_l \frac{d}{dt}[u(l,t)]^2 + \frac{1}{2}a_0 \frac{d}{dt}[u(0,t)]^2 = 0$$

The sum of the derivatives is the derivative of the sum.

$$\frac{d}{dt}\left\{E + \frac{1}{2}a_l[u(l,t)]^2 + \frac{1}{2}a_0[u(0,t)]^2\right\} = 0$$

Use the definition of E here.

$$\frac{d}{dt}\left\{\frac{1}{2}\int_0^l (c^{-2}u_t^2 + u_x^2)\,dx + \frac{1}{2}a_l[u(l,t)]^2 + \frac{1}{2}a_0[u(0,t)]^2\right\} = 0$$

Therefore,

$$E_R = \frac{1}{2}\int_0^l (c^{-2}u_t^2 + u_x^2)\,dx + \frac{1}{2}a_l[u(l,t)]^2 + \frac{1}{2}a_0[u(0,t)]^2$$

is conserved when there are Robin boundary conditions.